Conditionals as quotients: an algebraic approach

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Conditional expressions are central in representing knowledge and reasoning abilities of intelligent agents. Conditional reasoning indeed features in a wide range of areas including non-monotonic reasoning, causal inference, learning, and more generally reasoning under uncertainty. A conditional statement is a hypothetical proposition of the form "If [antecedent] is the case, then [consequent] is the case", where the antecedent is assumed to be true. Such a notion can be formalized by expanding the language of classical logic by a binary operator a/b that reads as "a given b". A most well-known approach in this direction is that of Stalnaker [4,5], further analyzed also by Lewis [2], that in order to axiomatize the operator / ground their investigation on particular Kripke-like structures. In this contribution we base our approach instead in the algebraic framework, in a line of investigation initiated in [1].

The novel approach we propose here is grounded in the algebraic setting of Boolean algebras, where there is a natural way of formalizing conditional statements. Indeed, given a Boolean algebra $\mathbf{B}$ and an element b in B, one can define a new Boolean algebra, say $\mathbf{B}/b$, intuitively obtained by assuming that b is true. More in details, one considers the congruence collapsing b and the truth constant 1, and then $\mathbf{B}/b$ is the corresponding quotient. Then the idea is to define a conditional operator / such that a/b represents the element a as seen in the quotient $\mathbf{B}/b$, mapped back to $\mathbf{B}$. The particular structural properties of Boolean algebras allow us to do so in a natural way.

First, we assume the algebra $\mathbf{B}$ to be finite. Then, if $b \neq 0$ the quotient $\mathbf{B}/b$ is actually a *retract* of $\mathbf{B}$, which means that if we call π_b the natural epimorphism $\pi_b : \mathbf{B} \to \mathbf{B}/b$, there is an injective homomorphism $\iota_b : \mathbf{B}/b \to \mathbf{B}$ such that $\pi_b \circ \iota_b$ is the identity map. The idea is then to consider

$$a/b := \iota_b \circ \pi_b(a). \tag{1}$$

We observe that the map ι_b is not uniquely determined, meaning that there can be different injective homomorphisms ι, ι' such that $\pi_b \circ \iota = \pi_b \circ \iota'$ is the identity. Now, in order to be able to define an operator / over the algebra $\mathbf{B}$, one needs to consider all the different quotients, determined by all choices of elements $b \in B$. Then, if $0 \neq b \leq c$, by general algebraic arguments one gets a natural way of looking at nested conditionals; indeed it holds that $(\mathbf{B}/c)/\pi_c(b) = \mathbf{B}/b$, which means that $\mathbf{B}/b$ is a quotient of $\mathbf{B}/c$, and actually also its retract. It is then natural to ask that the choices for ι_b and ι_c be *compatible*, in the sense that there is a way of choosing the embedding $\iota_{\pi_c(b)}$ so that

$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}, \tag{2}$$

which yields in particular that $a/b = (a/b)/c$ whenever $b \leq c$.

The case where $b = 0$ needs to be considered separately, since the associated quotient is the trivial algebra that cannot be embedded into $\mathbf{B}$. Since intuitively we are considering the quotients by an element b to mean that "b is true", the *ex falso quodlibet* suggests that we map all elements to 1, i.e:

$$a/0 := 1. \tag{3}$$

The idea is then to use Stone duality to translate the above conditions to the dual setting; in other words, we generate the intended models as algebras of sets.

To this end, by the finite version of Stone duality, we now see the algebra $\mathbf{B}$ as an algebra of sets, say that $\mathbf{B} = \mathcal{S}(X)$ for a set X. Then the above reasoning translates to the following. Given $Y \subseteq X$, the natural epimorphism $\pi_Y : \mathcal{S}(X) \to \mathcal{S}(Y)$ dualizes to the identity map $\mathrm{id}_Y : Y \to X$, and the embedding $\iota_Y : \mathcal{S}(Y) \to \mathcal{S}(X)$ dualizes to a surjective map $f_Y : X \to Y$, such that $f_Y \circ \mathrm{id}_Y = \mathrm{id}_Y$; in other words, we are asking that f_Y restricted to Y is the identity. Moreover, consider $Y \subseteq Z \subseteq X$. Then the compatibility condition (2) becomes on the dual $f_Y = f_Y^Z \circ f_Z$, where f_Y^Z is the dual of the map $\iota_{\pi_Z(Y)}$. The intended models are those that originate by the above postulates; let us be more precise.

Definition 1. *Given a set X, we say that a class of surjective functions $\mathcal{F} = \{f_Y^Z : Z \to Y : \emptyset \neq Y \subseteq Z \subseteq X\}$ with $f_Y^Z : Z \to Y$ is* compatible with X *if:*

1. *f_Y^X restricted to Y is the identity on Y;*
2. *$f_Y^X = f_Y^Z \circ f_Z^X$.*

We now define the class of intended models as algebras of sets.

Definition 2. *An* intended model *is an algebra with operations $\{\wedge, \vee, \neg, /, 0, 1\}$ that is a Boolean algebra of sets $\mathcal{S}(X)$ for some set X with $/$ defined as follows from a class of functions $\mathcal{F}$ compatible with X:*

$$Y/Z := (f_Z^X)^{-1}(Y \cap Z)$$

for any $Y, Z \subseteq X$ and $Z \neq \emptyset$, and for any $Y \subseteq X$ we set $Y/\emptyset := X$.

We then analyze the algebraic properties of the intended models and the variety they generate, showing also an interesting connection with Stalnaker's approach to conditionals.

References

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