Conditional Possibilities, Possibilistic Imaging and Boolean Algebras of Conditionals

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Abstract

A fundamental question that has been addressed at the intersection of knowledge representation and probability theory is: can we find a logical connective $\rhd$ whose probability aligns exactly with the corresponding conditional probability? More precisely, the question is if it possible to define a binary operator $\rhd$: $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ over a Boolean algebra $\mathbf{A}$ such that for all $a, b \in A$ with $b \neq \bot$

$$P(b \rhd a) = P(a \mid b) \quad (1)$$

This question delves beyond technical considerations, prompting a deeper examination of conditional probability's nature within knowledge representation. Specifically, it investigates whether conditional probability can be interpreted and reduced to the probability of a true conditional statement.

Initially, Stalnaker's conditional [11] appeared promising in fulfilling this role, see [12]. However, subsequent work by Lewis [8] and Hájek [6] demonstrated that no (truth-functional) conditional connective within the same Boolean algebra can fulfill this function without trivializing the probability function. These "triviality results" hold significant weight, revealing that conditional probability cannot be directly interpreted as the probability of a conditional relation definable within the original Boolean framework.

Building on these insights, [8] and [5] showed that the probability of a Stalnaker conditional $b \mathrel{\Box\!\!\rightarrow} a$ can be characterized using a more general update rule for probabilities, called "imaged probability":

$$P(b \mathrel{\Box\!\!\rightarrow} a) = P_b(a). \quad (2)$$

Here, $P_b(\cdot)$ represents a new probability measure, reflecting the scenario where b is true and resulting from transferring all the probability mass of where b is false to the relevant worlds where b is true. More precisely, a Stalnaker conditional, or equivalently a conditional in the logic C2 in [7], is interpreted with respect to a possible worlds model $\Sigma = (\Omega, \mathscr{S}, v)$ where $\mathscr{S}$ is a sphere system that provides a similarity ordering among worlds, and v is a valuation function; here

$b \,\Box\!\!\rightarrow a$ is deemed true at a world w whenever a (the consequent) is true in *the most similar* world where b (the antecedent) is true. Hence, P_b is the result of transferring the mass of each non-b-world to its most similar b-world; in order to perform the imaging update procedure, a similarity structure over our sample space is needed.

This imaging procedure has become a powerful alternative to conditionalization for updating probabilities with new information, e.g. [9]. Notably, it allows for an interpretation as the probability of a suitable conditional statement being true. For instance, [10], following [2], show how to extend the imaging procedure to Dempster-Shafer belief functions in order to characterize the probability of Lewis counterfactuals, i.e. conditionals in Lewis' logic C1 [7].

The present contribution explores similar questions within the framework of possibility theory [1], where "possibility measures" are a formal tool to represent uncertainty and knowledge which, unlike probability, rank events based on their plausibility. Interestingly, possibility theory also accounts for conditionalization and conditional possibility measures. Hence, analogous questions arise for possibility measures: is it possible to represent conditional possibilities as possibilities of conditionals? If not, how can we characterize the possibility of conditional logical operators?

We address these questions by proving first a new triviality result for possibility theory demonstrating that no truth-conditional operator in a Boolean setting can directly capture (a reasonable notion of) conditional possibilities. Building on this triviality result, we show that the possibility of well-known conditionals (like Stalnaker conditional and Lewis counterfactual) can be characterized using a generalization of "imaged possibility measures" introduced by Dubois and Prade in [2]. This establishes a deep connection between logical conditional operators and the imaging update procedure within possibility theory. We then assess our results against the work of [4]. They show how conditional possibilities can be represented as *canonically extended possibility measures* within the so-called *Boolean algebras of conditionals* (BACs) [3]. Their results allows us to represent the conditional operators we analysed, along with their induced imaged possibilities, in a highly expressive algebraic framework.

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References

1. Dubois, D., and Prade, H. 1988. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum Press.
2. Dubois, D., and Prade, H. 1994. A survey of belief revision and updating rules in various uncertainty models. *International Journal of Intelligent Systems* 9:61–100.
3. Flaminio, T.; Godo, L.; and Hosni, H. 2020. Boolean algebras of conditionals, probability and logic. *Artificial Intelligence* 286:102237.
4. Flaminio, T.; Godo, L.; and Ugolini, S. 2021. Canonical Extension of Possibility Measures to Boolean Algebras of Conditionals. In *Proceedings of Symbolic and Quantitative Approaches to Reasoning with Uncertainty - ECSQARU 2021*, 543–556. LNCS - Springer.
5. Gärdenfors, P. 1982. Imaging and conditionalization. *Journal of Philosophy* 79(12):747–760.
6. Hájek, A. 2011. Triviality pursuit. *Topoi* 30:3–15.
7. Lewis, D. 1971. Completeness and decidability of three logics of counterfactual conditionals. *Theoria* 37(1):74–85.
8. Lewis, D. 1976. Probabilities of conditionals and conditional probabilities. *The Philosophical Review* 85(3):297–315.
9. Pearl, J. 2017. Physical and metaphysical counterfactuals: Evaluating disjunctive actions. *Journal of Causal Inference* 5(2):20170018.
10. Rosella, G.; Flaminio, T.; and Bonzio, S. 2023. Counterfactuals as modal conditionals, and their probability. *Artificial Intelligence* 323:103970.
11. Stalnaker, R. 1968. A theory of conditionals. In Rescher, N., ed., *Studies in Logical Theory (American Philosophical Quarterly Monographs 2)*. Blackwell. 98–112.
12. Stalnaker, R. C. 1970. Probability and conditionals. *Philosophy of Science* 37(1):64–80.