MTL does not have the interpolation property

Valeria Giustarini and Sara Ugolini

IIIA, CSIC, Barcelona, Spain

Mathematical Fuzzy Logic (or MFL) was introduced by Hájek in 1998 as a mathematical logic framework with the intention of modeling logical reasoning with vague statements. The *monoidal t-norm based logic* MTL, introduced by Esteva and Godo in [2], belongs to the MFL framework and it is the logic of left-continuous t-norms and their residua. This logic has an equivalent algebraic semantics given by MTL-algebras, that are particular *residuated lattices*, i.e. lattice-ordered monoids with an implication. Importantly, the variety of MTL-algebras is generated by totally ordered structures, and in particular by all algebras with domain over the real-unit interval $[0, 1]$ obtained by considering the monoidal operation to be a left-continuous t-norm.

In this work we tackle a long-standing open problem for MTL. To the best of our knowledge, it was not known whether MTL satisfied the deductive interpolation property or not; we will answer in the negative. We say that a logic L, associated to an equivalence relation $\vdash$, has the *deductive interpolation property* if for any set of formulas $\Gamma \cup \{\psi\}$ over the appropriate language, if $\Gamma \vdash \psi$ then there exists a formula δ such that $\Gamma \vdash \delta$, $\delta \vdash \psi$ and the variables appearing in δ belong to the intersection of the variables appearing both in Γ and in ψ, in symbols $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$. In order to give an answer to this problem, we move to the algebraic setting using a deep *bridge theorem* which states that the interpolation property of a (strongly algebraizable) logic corresponds to the amalgamation property of the corresponding variety of algebras, under the assumption that such variety has the congruence extension property (CEP). To be more precise, the amalgamation property for a variety is equivalent to the Robinson property for its associated logic; the latter implies the deductive interpolation property, which implies the amalgamation property in the presence of the CEP. Hence, since MTL satisfies all the needed properties, in our contribution we study the amalgamation property in the variety of MTL-algebras, by means of some new algebraic constructions.

In order to obtain a failure of the amalgamation property, we use the recent results in [4]; the authors show that in a variety V with the congruence extension property (CEP) and whose class of finitely subdirectly irreducible members $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras, the amalgamation property (or AP) of the variety is equivalent to the so-called *one-sided amalgamation property* (1AP for short) of $\mathsf{V}_{\mathrm{FSI}}$. This result is particularly useful in subvarieties of MTL-algebras, that have the CEP and whose finitely subdirectly irreducibles are exactly all the totally ordered algebras.

In order to give more details, let us recall the necessary definitions. Given a class K of algebras in the same signature, a *V-formation* is a tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$ respectively;

an *amalgam* in K for the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ is a triple $(\mathbf{D}, h, k)$ where $\mathbf{D} \in \mathsf{K}$, and h and k are embeddings of respectively $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$ such that $h \circ i = k \circ j$. A *one-sided amalgam* is instead a triple $(\mathbf{D}', h', k')$ with $\mathbf{D}' \in \mathsf{K}$ and again $h' \circ i = k' \circ j$ but while h' is an embedding, k' is a homomorphism. A class K of algebras has the (one-sided) amalgamation property if for any V-formation there is a (one-sided) amalgam in K.

We exhibit a V-formation, that we call $\mathcal{VS}$-formation, given by 2-potent MTL-chains, that does not have an amalgam in the class of totally ordered (equivalently, finitely subdirectly irreducible) MTL-algebras. This yields the failure of the one-sided amalgamation property in the class of MTL-chains. More precisely, using the result in [4] we can prove the following theorem.

Theorem 1. *Let V be a variety of semilinear residuated lattices with the congruence extension property, and such that the algebras in $\mathcal{VS}$ belong to V. Then V does not have the amalgamation property.*

Hence, we get the following relevant examples of the above result.

Corollary 1. *The following varieties do not have the amalgamation property:*

1. *Semilinear commutative residuated lattices;*
2. *Semilinear commutative integral residuated lattices;*
3. *Semilinear commutative residuated lattices with an extra constant 0;*
4. *MTL-algebras;*
5. *n-potent MTL-algebras for $n \geq 2$.*

Thus, as a consequence of the previous result we get in particular:

Corollary 2. *MTL does not have the deductive interpolation property.*

In order to understand the algebras involved in the $\mathcal{VS}$-formation, we develop some new algebraic constructions that allow to construct new chains from known ones, which generalize the *partial gluing construction* introduced in [5].

References

1. Blok W., Pigozzi D.: *Algebraizable Logics*, Mem. Amer. Math. Soc, 396(77). Amer. Math Soc. Providence,1989.
2. Esteva F., Godo L.: *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy sets and systems 124(3), 271–288, 2001.
3. Galatos N., Jipsen P., Kowalski T., Ono H.: *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.
4. Fussner W., Metcalfe G.: Transfer theorems for finitely subdirectly irreducible algebras,Journal of Algebra,Volume 640,2024,Pages 1-20,ISSN 0021-8693,
5. Galatos, N., Ugolini, S.: Gluing Residuated Lattices. Order 40, 623–664 (2023).