Precompact Fuzzy Indistinguishability Relations: A Model of Bounded Discernment in Similarity-Based Reasoning

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Abstract. Precompact similarity relations are proposed to model limited discernment abilities of resource-bounded agents in similarity-based reasoning. A few corollaries and applications of the principle of precompactness are sketched.

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Fuzzy indistinguishability relations, also known as fuzzy similarity relations or fuzzy equivalence relations (i.e., fuzzy relations that are reflexive, symmetric, and T-transitive for a given t-norm T), are a well established model of the phenomena of similarity and indistinguishability [7,5]. As a model of indistinguishability, fuzzy equivalences enjoy several convenient properties, such as the duality to (generalized pseudo-) metrics, easy treatment by means of formal fuzzy logic, elimination of Poincaré's paradox, and several kinds of representation theorems. Various special classes of fuzzy indistinguishability relations have been studied, often delimited by the corresponding metric properties, such as boundedness, length, or 1-dimensionality [4].

To model bounded discernment abilities of real-world agents, I propose a hitherto unconsidered property of fuzzy indistinguishability relations and study some of its corollaries. The property embodies the principle that bounded agents can never fully distinguish all elements of an infinite set (cf. Vopěnka's nonstandard treatment of infinity, [6]). If we apply the latter principle to the model of indistinguishability in fuzzy mathematics, it amounts to the following condition on fuzzy indistinguishability relations:

Definition 1. *Let R be a fuzzy indistinguishability relation on a domain X. We say that R is* precompact *if the following condition holds for all infinite $A \subseteq X$:*

$$\bigvee_{\substack{a,b \in A \\ a \neq b}} Rab = 1. \tag{1}$$

The name of the condition is motivated by the fact that it is equivalent to the requirement of *metric precompactness* (i.e., total boundedness) of the generalized pseudometric dual to the fuzzy indistinguishability relation R.

Example 1. An example of a precompact fuzzy indistinguishability relation (T-transitive w.r.t. the Łukasiewicz t-norm) on $\mathbb{R}$ is

$$Rxy = 1 - \tfrac{1}{\pi}|\arctan x - \arctan y|. \tag{2}$$

The Euclidean similarity $Exy = \max\big(|x-y|, 0\big)$ is not precompact on $\mathbb{R}$, although it is precompact on any bounded interval $[a, b]$ for $a, b \in \mathbb{R}$.

Corollary 1. *If R is a precompact fuzzy indistinguishability on X and $A \subseteq X$ an infinite set, then for all $\alpha < 1$ there are $a, b \in A$ such that $Rab > \alpha$.*

Thus under a precompact indistinguishability, only finitely many elements can be distinguished from one another at least to degree α, for each $\alpha < 1$ (but the number of α-distinguishable elements can increase with $\alpha \nearrow 1$).

The condition of precompactness can easily be expressed by means of formal fuzzy logic and investigated by the methods of formal fuzzy mathematics: by the semantics of t-norm fuzzy logics [3], condition (1) can be expressed by the formula $(\exists a \in A)(\exists b \in A)(a \neq b \;\&\; Rab)$, where & is the strong conjunction interpreted by a left-continuous t-norm T. (Recall that in t-norm fuzzy logics, the quantifiers $\exists$ and $\forall$ are interpreted as the supremum and the infimum, respectively, and the implication $\rightarrow$ as the residuum of the t-norm T; see [3] for details.) To facilitate the formulation of further corollaries to precompactness, let us denote the truth degree by a formula φ in the fuzzy logic based on a t-norm T by $\|\varphi\|_T$.

The precompactness of a fuzzy indistinguishability relation brings about many properties analogous to those of compact metric spaces, such as the following ones:

Theorem 1. *Given a precompact fuzzy indistinguishability $\sim$ on $\mathbb{R}$, every function $f\colon \mathbb{R} \to \mathbb{R}$ is* bounded modulo $\sim$, *i.e.,*

$$\big\|(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})\big((y \geq f(x)) \vee (y \sim f(x))\big)\big\|_T = 1. \tag{3}$$

Theorem 2. *Let $\preceq$ be a fuzzy ordering on X that fuzzifies a crisp ordering $\leq$ on X by a $\leq$-compatible fuzzy indistinguishability relation $\sim$ on X. If $\sim$ is precompact, then every non-empty fuzzy subset A of X has a non-empty fuzzy minimum in $\preceq$, i.e.,*

$$\big\|(\exists x \in X)\big(Ax \wedge (\forall y \in X)(Ay \rightarrow (x \preceq y))\big)\big\|_T > 0. \tag{4}$$

Proof. Theorem 1 follows easily from Cor. 1. For Th. 2 cf. [1], where precompactness ensures (suitably modified) preconditions of the main theorems. □

Further corollaries of the precompactness principle are, for instance, the piecewise-linear approximability and the computability of all real functions, modulo precompact fuzzy indistinguishability. Note, however, that similarly to (3) and (4), these corollaries are formulated by means of a t-norm fuzzy logic. This makes them weaker than analogous claims formulated in classical logic (which would be false), though often sufficiently strong for modeling bounded agents' perception and reasoning (as seen from an application referenced below).

The mentioned properties of fuzzy precompact indistinguishability relations can be applied in similarity-based reasoning that takes the limited discernment abilities of real-world agents into account. An example is the use of (a variant of) Th. 2 in the recently proposed fuzzy semantics of counterfactual conditionals [2], where the artificial-looking condition of the right-connectedness of the similarity of possible worlds can be justified as a consequence of its precompactness, i.e., a natural consequence of the limited ability of bounded reasoners to fully distinguish all of the differences between infinitely many possible worlds. The resulting existence of fuzzy minima makes it possible to avoid the implausible Limit Assumption in the original Stalnaker–Lewis semantics of counterfactuals (see [2] for details).

A further exploration of the applicability of the precompactness principle in similarity-based reasoning and the comprehensive presentation of the notion is a work in progress.

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