Ordinal sum of semigroups yielding a uninorm

Martin Kalina[0000−0002−8072−1574]

Slovak University of Technology, Faculty of Civil Engineering
Radlinského 11, SK-810 05 Bratislava, Slovakia,
martin.kalina@stuba.sk

Abstract. By now, some generalizations of the ordinal sum construction by Clifford were studied where the summands were some t-subnorms and/or t-superconorms. The main property of all t-subnorms as well as of t-superconorms used in the ordinal sum constructions, is the existence of (trivial) idempotent elements. Particularly, 0 and 1 are idempotents of t-subnorms and t-superconorms, respectively. When considering suitable monotone commutative semigroups as the summands in the ordinal sum construction yielding a uninorm, these semigroups may well be without any idempotent elements. It will be shown that the existence of idempotent elements is not crucial for the ordinal sum construction. In the contribution we will study the properties of uninorms constructed via such ordinal sums.

Keywords: idempotent element, ordinal sum, commutative monotone semigroup, uninorm

1 Introduction and Some Known Facts

Ordinal sums of semigroups were proposed by Clifford [1]. They are a possibility how to construct a new t-norm (t-conorm) from given ones (see [5]). De Baets and Mesiar [2] generalized the ordinal sum constructions for arbitrary aggregation functions. Another important step in developing ordinal sum constructions was made by Mesiarová-Zemánková [6]. She generalized the ordinal sum constructions for uninorms proposing the z-ordinal sum. The paper [6], together with results in the author's paper [3], are the main motivation for the present research.

The most important result in [3], for this research, is the following

Theorem 1 ([3]). *Let $\tilde{T}_{0,a}$ be a t-subnorm having an idempotent element 1, and $\tilde{T}_{a,1}$ be a t-norm and $a \in \,]0,1[$ be arbitrarily chosen. Further, denote $T_{0,a}$ the isomorphic linear transform of $\tilde{T}_{a,2}$ into $[0,a]$ and $T_{a,1}$ the isomorphic linear transform of $\tilde{T}_{a,1}$ into $[a,1]$. Then*

$$T(x,y) = \begin{cases} T_{0,a}(x,y) & \text{for } (x,y) \in [0,a]^2, \\ T_{a,1}(x,y) & \text{for } (x,y) \in [a,1]^2, \\ T_{0,a}(x,a) & \text{for } x \in [0,a],\ y \in \,]a,1[, \\ T_{0,a}(a,y) & \text{for } y \in [0,a],\ x \in \,]a,1[, \\ \min(x,y) & \text{if } max(x,y) = 1, \end{cases}$$

is a t-norm.

The construction in Theorem 1 pis said to be a generalized ordinal sum. It can be straightforwardly generalized for an ordinal sum of countably many t-subnorms $\{T_{a_{i-1},a_i}\}_{i=1}^{\infty}$ (for which the top element is idempotent). It will be denoted in the following way

$$T = \hat{\oplus}_{i=1}^{\infty} \langle T_i, a_{i-1}, a_i \rangle, \tag{1}$$

where $\{a_i\}_i$ is an increasing sequence (not necessarily strictly increasing) such that $a_0 = 0$ and $a_\infty = 1$. Dually, the ordinal sum of t-superconorms $\{S_{b_i,b_{i-1}}\}_{i=1}^{\infty}$ can be introduced, where $\{b_i\}_i$ is a decreasing sequence (not necessarily strictly increasing) such that $b_0 = 1$ and $b_\infty = 0$.

An ordinal sum construction for uninorms, where the "building bricks" are t-subnorms and t-superconorms, was presented by the author at FSTA 2024 [4]. After the presentation, the author got a question posted by Paweł Drygaś, namely, what happens if the top element of the t-subnorms (the bottom element of the t-superconorms) is not idempotent. With this question, the present research has started.

2 Main Results

This section will be started with (by now unpublished) results presented at FSTA 2024. Set

$$A_e = \,]0,e[\, \times \,]e,1[\, \cup \,]e,1[\, \times \,]0,e[\,.$$

Theorem 2. *Let U be a uninorm and $e \in \,]0,1[$ its neutral element. Denote $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ the respective monotone sequences of idempotent elements of U in $[0,e]$ and $[e,1]$, respectively, such that $a_0 = 0$ and $b_0 = 1$.*

(P) *Assume for every $a_i \in \,]0,e[$ the uninorm U restricted to $[0,a_i]$ is a proper t-subnorm, and for every $b_i \in \,]e,1[$ the unonorm U restricted to $[b_i,1]$ is a proper t-superconorm.*

Then, if there exist a pair $(x,y) \in A_e$ such that $U(x,y) \in \,]0,e[$, then

$$U(x,y) \in \,]0,e[\quad \text{for all } (x,y) \in A_e.$$

Theorem 3. *Let U be a uninorm and $e \in \,]0,1[$ its neutral element. Assume Property (P) from Theorem 2 holds for U. Denote*

$$\Theta_i = \{x \in [0,e]; U(x,a_i) = x\}, \tag{2}$$

where $\{a_i\}_{i=0}^{\infty}$ is the monotone sequence of idempotents of U in $[0,e]$. Assume $U(x,y) \in \,]0,e[$ for $(x,y) \in A_e$. Then $U(x,y) = x$ for all $x \in \Theta_i$ and $y \in [a_i,1]$.

Theorem 4. *Let U be a uninorm and $e \in \,]0,1[$ its neutral element. Assume Property (P) from Theorem 2 holds for U. Let the sets Θ_i be defined by formula*

(2). *Denote $I \subset [0,e]$ a closed interval such that $I \cap \Theta_i = \emptyset$ for all $i \in \mathbb{N}$. Assume U restricted to I is constant. For a $j \in \mathbb{N}$ let $b_j < b_{j-1}$. Set $\varphi : i \to [b_j, b_{j-1}]$ a monotone bijection and let S_j denote the restriction of U to $[b_j, b_{j-1}]$. Then for $x \in I$ and $y \in [b_j, b_{j-1}]$ one may have*

$$U(x,y) = \varphi^{-1}\left(S(\varphi(x), y)\right).$$

Of course, Theorems 2 up to 4 can be reformulated for the case $U(x,u) > e$ for $(x,y) \in A_e$.

Now, Theorems 2, 3 and 4 will be reformulated for the case that instead of t-subnorms and t-superconorms with at least two idempotent elements only commutative monotone semigroups, possibly without idempotent elements, will be considered.

Example 1. Let T (S) be a t-norm without 0-divisors (a t-conorm without 1-divisors) and without any idempotents. Then T restricted to $]0,c]$ (S restricted to $[c,1[$) is a monotone commutative semigroup without idempotent elements.

The ordinal sum construction $\hat{\oplus}_{i=1}^{\infty}\langle T_i, a_{i-1}, a_i\rangle$ introduced by formula (1) will be modified. The intervals with endpoints a_{i-1} and a_i will be chosen correspondingly to T_i to yield a semigroup.

Theorem 5. *Let U be a uninorm and $e \in\ ]0,1[$ be its neutral element. Let $\hat{\oplus}_{i=1}^{\infty}\langle T_i, A_i\rangle$ be its underlying t-norm, where the summands, T_i, be monotone commutative semigroups. Denote a_i the right endpoint of A_i. If for an $x \in A_i$ $U(x,y_1) = U(x,y_2) = c < x$ for $y_1, y_2 \in\ ]a_i, e[$, then*

(C1) *$U(z,y) = c$ for all $z \in [c,x]$ and $y \in\ ]a_i, e[$,*
(C2) *$U(c,z_1) = U(c,z_2)$ for all $z_1, z_2 \in [c,x]$.*

Remark 1 (to Theorem 5). The property (C2) implies that if U restricted to an interval A_i is a cancelative t-norm then $U(x,y) = x$ for all $x \in A_i$ and all $y \in\ ]a_i, e]$ where a_i is the right endpoint of A_i.

A possibility how a uninorm U may look like is presented in the following example.

Example 2. Set $a_1 = \frac{1}{4}$, $e = \frac{1}{2}$, $b_1 = \frac{3}{4}$. The underlying t-norm of the just constructing uninorm is given by the following formula

$$U(x,y) = \begin{cases} max(0, x+y-\frac{3}{8}) & \text{for } x,y) \in [0,a_1]^2, \\ (x-\frac{1}{4})(y-\frac{1}{4}) + \frac{1}{4} & \text{for } (x,y) \in\ ]a_1, e[, \\ 0 & \text{if } \min(x,y) < \frac{1}{8} \text{ and } \max(x,y) \in\ ]a_1, e[, \\ \frac{1}{8} & \text{if } \min(x,y) \in [\frac{1}{8}, \frac{1}{4}] \text{ and } \max(x,y) \in\ ]a_1, e[, \\ \min(x,y) & \text{if } \max(x,y) = e. \end{cases}$$

For simplicity reasons, let the underlying t-conorm of U be dual to the underlying t-norm. For $(x,y) \in A_e$ one may take min. The U has just three idempotent elements, namely $0, e, 1$.

3 Conclusions

In this short paper some generalized ordinal sum construction for uninorms was presented and and basic properties were presented. One example showing that such uninorm may possess only trivial idempotent elements.

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