On generalized n-dimensional grouping indices

Tiago Asmus[1][0000-0002-7066-7156], Humberto Bustince[2][0000-0002-1279-6195], Giancarlo Lucca[33][0000-0002-3776-0260], Cedric Marco-Detchart[4][0000-0002-4310-9060], Heloisa Camargo[5][0000-0002-5489-7306], Joelson Sartori[1][0000-0001-5379-8253], and Graçaliz Dimuro[1][0000-0001-6986-9888]

[1] Universidade Federal do Rio Grande, Rio Grande 96201-900, Brazil
tiagoasmus@furg.br
[2] Universidad Pública de Navarra, Pamplona 31006, Spain
[3] Universidade Católica de Pelotas, Pelotas 96015-560, Brazil
[4] Universitat Politècnica de València, Valencia 46022, Spain
[5] Universidade Federal de São Carlos, São Carlos 13565-905, Brazil

Abstract. The concept of grouping index has been recently introduced as a comparison index $\mathcal{G}$ of two fuzzy subsets A and B that provides a measure of the proximity of the *fuzzy union* of A and B to the *referential set* U, in such a way that "the higher $\mathcal{G}(A, B)$ is, the closer to U $A \cup B$ is." Grouping indices can be seen as the dual notion of overlap indices, and both can be constructed by aggregation of grouping and overlap functions, respectively. However, grouping indices are bivariate functions, limiting their application to the comparison of two fuzzy sets at a time. Inspired by the recent development of n-dimensional and general overlap indices, in this work we expand the concept of grouping indices to the n-dimensional context, introducing n-dimensional grouping indices, and also generalize this new concept by allowing a less restrictive definition, which bring us the concept of general grouping indices. We pay special attention to the development of construction methods for these new generalized grouping indices.

Keywords: general grouping functions · overlap indices · grouping indices

1 Introduction

Overlap indices are functions that can act as a measure of the fuzzy intersection between two fuzzy sets. Recently, new definitions of overlap indices have been introduced: one in which allow them to measure the overlap between n fuzzy sets, called n-dimensional overlap indices, and another generalizes both previous concepts by allowing less restrictive conditions, called general overlap indices [1].

Although overlap indices have been studied for many years, only recently the dual notion of them has been introduced, called grouping indices [4]. As expected, grouping indices can act as measure of the fuzzy union between two fuzzy sets. One key aspect of both overlap and grouping indices is that they can be constructed by an aggregation of overlap and grouping functions, respectively. In the same manner, general overlap indices can also be constructed by an aggregation of general overlap functions [1].

However, the development of grouping indices is still in its early stages, as they were only defined as bivariate functions, limiting their application to the comparison of two fuzzy sets at a time. So, inspired in the development of generalized overlap indices, the objective of this work is to expand the concept of grouping indices to the n-dimensional context, introducing n-dimensional grouping indices, and also to generalize this new concept by allowing a less restrictive definition, introducing the concept of general grouping indices. We are going to pay special attention to the development of construction methods for these generalized grouping indices.

2 Preliminary concepts

A function $N : [0, 1] \to [0, 1]$ is a fuzzy negation [6] if the following conditions hold: **(N1)** $N(0) = 1$ and $N(1) = 0$; **(N2)** If $x \leq y$ then $N(y) \leq N(x)$, for all $x, y \in [0, 1]$. If N also satisfies **(N3)** $N(N(x)) = x$, for all $x \in [0, 1]$, then it is said to be a strong fuzzy negation. An aggregation function [2] is any function $A : [0, 1]^n \to [0, 1]$ such that: **(A1)** A is increasing; **(A2)** $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

A function $On : [0, 1]^n \to [0, 1]$ is said to be an n-dimensional overlap function [5] if, for all $\mathbf{x} \in [0, 1]^n$: **(On1)** On is commutative; **(On2)** $On(\mathbf{x}) = 0 \Leftrightarrow \prod_{i=1}^n x_i = 0$; **(On3)** $On(\mathbf{x}) = 1 \Leftrightarrow \prod_{i=1}^n x_i = 1$; **(On4)** On is increasing; **(On5)** On is continuous.

A function $GO : [0, 1]^n \to [0, 1]$ is said to be a general overlap function [3] if respects, for all $\mathbf{x} \in [0, 1]^n$, conditions **(On1)**, **(On4)**, **(On5)** and: **(GO2)** If $\prod_{i=1}^n x_i = 0$ then $GO(\mathbf{x}) = 0$; **(GO3)** If $\prod_{i=1}^n x_i = 1$ then $GO(\mathbf{x}) = 1$.

A function $Gn : [0, 1]^n \to [0, 1]$ is said to be an n-dimensional grouping function [5] if, for all $\mathbf{x} \in [0, 1]^n$: **(Gn1)** Gn is symmetric; **(Gn2)** $Gn(\mathbf{x}) = 0 \Leftrightarrow x_i = 0$ for all $i \in \{1, \ldots, n\}$; **(Gn3)** $Gn(\mathbf{x}) = 1 \Leftrightarrow$ there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$; **(Gn4)** Gn is increasing; **(Gn5)** Gn is continuous.

A function $GG : [0, 1]^n \to [0, 1]$ is said to be a general grouping function [7] if, for all $\mathbf{x} \in [0, 1]^n$, conditions hold: **(Gn1)**, **(Gn4)**, **(Gn5)** and: **(GG2)** If $x_i = 0$ for all $i \in \{1, \ldots, n\}$, then $GG(\mathbf{x}) = 0$; **(GG3)** If there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$ then $GG(\mathbf{x}) = 1$.

One can construct general grouping functions as truncated versions of n-dimensional grouping functions, as follows:

Proposition 1. *[3] Take $\alpha, \beta, \epsilon, \delta \in [0, 1]$, such that $\alpha + \epsilon \leq \beta - \delta \leq 1$, and let $Gn : [0, 1]^2 \to [0, 1]$ be an n-dimensional grouping function. Then, the function $GTGn_{\alpha,\beta}^{\epsilon,\delta} : [0, 1]^n \to [0, 1]$, defined, for all $\mathbf{x} \in [0, 1]^n$, by*

$$GTGn_{\alpha,\beta}^{\epsilon,\delta}(\mathbf{x}) = \begin{cases} 0, & \text{if } Gn(\mathbf{x}) \leq \alpha \\ \frac{\alpha+\epsilon}{\epsilon}\left(Gn(\mathbf{x}) - \alpha\right), & \text{if } \alpha < Gn(\mathbf{x}) < \alpha + \epsilon \\ Gn(\mathbf{x}), & \text{if } \alpha + \epsilon \leq Gn(\mathbf{x}) \leq \beta - \delta \\ \beta - \delta + \frac{1-(\beta-\delta)}{\delta}\left(Gn(\mathbf{x}) - (\beta - \delta)\right), & \text{if } \beta - \delta < Gn(\mathbf{x}) < \beta \\ 1, & \text{if } Gn(\mathbf{x}) \geq \beta, \end{cases} \quad (1)$$

is a general grouping function.

We denote by $FS(U)$ the space of all fuzzy sets defined over $U = \{u_1, \ldots, u_m\}$.

Definition 1. *A mapping $\mathcal{O}n \colon FS(U)^n \to [0, 1]$ is said to be an overlap index (OvI) if it satisfies, for all $A_1, \ldots, A_n, B \in FS(U)$: ($\mathcal{O}$**n1**) $\mathcal{O}n(A_1, \ldots, A_n) = 0$ if and only* *if, for all* $u \in U, \prod_{i=1}^{n} A_i(u) = 0$*;* $(\mathcal{O}\mathbf{n2})$ $\mathcal{O}n$ *is commutative;* $(\mathcal{O}\mathbf{n3})$ *If* $A_1 \leq B$*, then* $\mathcal{O}n(A_1, A_2, \ldots, A_n) \leq \mathcal{O}n(B, A_2, \ldots, A_n)$*. For an* n*-dimensional OvI to be called normal, it also has to satisfy the following condition:* $(\mathcal{O}\mathbf{n4})$ *If there exists* $u \in U$ *such that* $\prod_{i=1}^{n} A_i = 1$*, then* $\mathcal{O}n(A_1, \ldots, A_n) = 1$*.*

Definition 2. *A mapping* $\mathcal{GO}\colon FS(U)^n \to [0,1]$ *is said to be a general overlap index (GOvI) if it satisfies, for all* $A_1, \ldots, A_n, B \in FS(U)$*:* $(\mathcal{GO}\mathbf{1})$ *If, for all* $u \in U, \prod_{i=1}^{n} A_i = 0$*, then* $\mathcal{GO}(A_1, \ldots, A_n) = 0$*;* $(\mathcal{GO}\mathbf{2})$ $\mathcal{GO}$ *is commutative;* $(\mathcal{GO}\mathbf{3})$ *If* $A_1 \leq B$*, then* $\mathcal{GO}(A_1, A_2, \ldots, A_n) \leq \mathcal{GO}(B, A_2, \ldots, A_n)$*. For an* n*-dimensional OvI to be called normal, it also has to satisfy the following condition:* $(\mathcal{GO}\mathbf{4})$ *If there exists* $u \in U$ *such that* $\prod_{i=1}^{n} A_i = 1$*, then* $\mathcal{GO}(A_1, \ldots, A_n) = 1$*.*

Definition 3. *A mapping* $\mathcal{G}\colon FS(U)^2 \to [0,1]$ *is a grouping index (GrI) if, for all* $A_1, A_2, A_3 \in FS(U)$*:* $(\mathcal{G}\mathbf{1})$ $\mathcal{G}(A_1, A_2) = 1$ *if and only if, for all* $u \in U$*,* $A_1(u) = 1$ *or* $A_2(u) = 1$*;* $(\mathcal{G}\mathbf{2})$ *If* $A_2 \leq A_3$ *then* $\mathcal{G}(A_1, A_2) \leq \mathcal{G}(A_1, A_3)$*;* $(\mathcal{G}\mathbf{3})$ $\mathcal{G}(A_1, A_2) = \mathcal{G}(A_2, A_1)$*. For a GrI to be called anti-normal, it also has to satisfy the following condition:* $(\mathcal{G}\mathbf{4})$ *If there exists* $u \in U$ *such that* $A_1(u) = A_2(u) = 0$*, then* $\mathcal{G}(A_1, A_2) = 0$*.*

3 General grouping indices

Our first goal is to expand the concept of grouping indices to the n-dimensional context, as functions that measure the "fuzzy union" between n fuzzy sets:

Definition 4. *A mapping* $\mathcal{G}n : FS(U)^n \to [0,1]$ *is said to be an* n*-dimensional grouping index if it satisfies, for all* $A_1, \ldots, A_n, B \in FS(U)$*:* $(\mathcal{G}\mathbf{n1})$ $\mathcal{G}n(A_1, \ldots, A_n) = 1$ *if and only if, for all* $u \in U, A_i(u) = 1$ *for some* $i \in \{1, \ldots, n\}$*;* $(\mathcal{G}\mathbf{n2})$ $\mathcal{G}n$ *is commutative;* $(\mathcal{G}\mathbf{n3})$ *If* $A_1 \leq B$*, then* $\mathcal{G}n(A_1, A_2, \ldots, A_n) \leq \mathcal{G}n(B, A_2, \ldots, A_n)$*. For an* n*-dimensional GrI to be called anti-normal, it also has to satisfy the following condition:* $(\mathcal{O}\mathbf{n4})$ *If there exists* $u \in U$ *such that* $A_1(u) = \ldots = A_n(u) = 0$*, then* $\mathcal{G}n(A_1, \ldots, A_n) = 0$*.*

It is immediate that an 2-dimensional GrI is just a GrI, as in Def. 3. Most properties and construction methods for n-dimensional GrIs follow naturally from the ones in [4] for GrIs. So now, let us generalize n-dimensional GrIs by loosening condition $(\mathcal{G}\mathbf{n1})$:

Definition 5. *A mapping* $\mathcal{GG} : FS(U)^n \to [0,1]$ *is said to be a general grouping index (GGrI) if it satisfies, for all* $A_1, \ldots, A_n, B \in FS(U)$*, conditions* $(\mathcal{G}\mathbf{n2})$*,* $(\mathcal{G}\mathbf{n3})$ *and:* $(\mathcal{GG}\mathbf{1})$ *If, for all* $u \in U, A_i(u) = 1$ *for some* $i \in \{1, \ldots, n\}$*, then* $\mathcal{G}n(A_1, \ldots, A_n) = 1$*; For a GGrI to be called normal, it also has to satisfy condition* $(\mathcal{G}\mathbf{n4})$*.*

It is immediate that any n-dimensional grouping index is also a GGrI, but the converse does not hold. Now, we present some construction methods of GGrI:

Theorem 1. *Consider an aggregation function* $M : [0,1]^m \to [0,1]$ *with the condition "**(M1):*** $M(x_1, \ldots, x_m) = 1 \Leftrightarrow x_1 = \ldots = x_m = 1$*" and an* m*-uple of general grouping functions* $GG_1, \ldots, GG_m : [0,1]^n \to [0,1]$*. Then, the function* $\mathcal{GG}_M^{GG_1,\ldots,GG_m} : FS(U)^n \to [0,1]$*, given, for all* $A_1, \ldots, A_n \in FS(U)$*, by*

$$\mathcal{GG}_M^{GG_1,\ldots,GG_m}(A_1,\ldots,A_n)=M(GG_1(A_1(u_1),\ldots,A_n(u_1)),\ldots,GG_n(A_1(u_m),\ldots,A_n(u_m))),$$

is a general grouping index.

Thus, one can obtain general grouping indices based on choices of an aggregation M that respects **(M1)** (such as n-dimensional overlap functions) and general grouping functions $GG_1, \ldots, GG_m$, which, in turn, can be constructed via an n-dimensional grouping function $Gn_1, \ldots, Gn_m$ via Prop. 1. This allows the construction of very flexible general grouping indices, which can be adapted according to the application.

Theorem 2. *Consider an aggregation function $M : [0,1]^m \to [0,1]$ with the condition "**(M2)**: $M(x_1, \ldots, x_m) = 1 \Rightarrow \max\{x_1, \ldots, x_m\} = 1$" and an m-uple of general grouping indices $\mathcal{GG}_1, \ldots, \mathcal{GG}_m : FS(U)^n \to [0,1]$. Then, the function $\mathcal{GG}_M^{\mathcal{GG}_1,\ldots,\mathcal{GG}_m} : FS(U)^n \to [0,1]$, given, for all $A_1, \ldots, A_n \in FS(U)$, by*

$$\mathcal{GG}_M^{\mathcal{GG}_1,\ldots,\mathcal{GG}_m}(A_1, \ldots, A_n) = M(\mathcal{GG}_1(A_1, \ldots, A_n), \ldots, \mathcal{GG}_m(A_1, \ldots, A_n)),$$

is a general grouping index.

Thus, the composition of general grouping indices by an n-dimensional grouping function results in a GGrI, since such functions respect **(M2)**. Next, we construct a GGrI by duality from a GOvI:

Theorem 3. *Let $\mathcal{GO}\colon FS(U)^2 \to [0,1]$ be a GOvI and $N\colon [0,1] \to [0,1]$ be a strong fuzzy negation. Then, the fuzzy set function $\mathcal{GG}_N^{\mathcal{GO}} : FS(U)^2 \to [0,1]$, defined, for all $A, B \in FS(U)$, by $\mathcal{GG}_N^{\mathcal{GO}}(A,B) = N(\mathcal{GO}(N(A), N(B)))$ is a GGrI, called the N-dual of $\mathcal{GO}$.*

4 Conclusion

In this work, we present some generalizations on the concept of grouping indices, first by considering the measure of fuzzy union between n fuzzy sets and then by relaxing the condition in which the index may return the value one. We focused our attention on different construction methods for these generalizes grouping indices, by composition of general overlap/grouping functions or by duality with general overlap indices.

Acknowledgments. Supported by CNPq (304118/2023-0, 407206/2023-0), FAPESP (2022/ 09136-1), MCIN/AEI/10.13039/ 50100011033/FEDER, ERDF, UE (PID2021-123673OB-C31, PID2022-136627NB-I00).

References

1. Asmus, T., Santos, H., Dimuro, G., Camargo, H., Marco-Detchart, C., Bustince, H.: On generalized overlap functions and overlap indices. In: FLINS-ISKE 2024. (to appear)
2. Beliakov, G., Bustince, H., Calvo, T.: A Practical Guide to Averaging Functions. Springer, NY (2016)
3. De Miguel, L., Gomez, D., Rodríguez, J.T., Montero, J., Bustince, H., Dimuro, G.P., Sanz, J.A.: General overlap functions. Fuzzy Sets and Systems **372**, 81 – 96 (2019).
4. Dimuro, G., Santos, H., Urio-Larrea, A., Asmus, T., Lucca, G., Camargo, H., Parodi, M.E., Bustince, H.: Grouping indices: definition, properties and construction methods. In: FUZZ-IEEE/IEEE WCCI 2024. (to appear)
5. Gomez, D., Rodríguez, J.T., Montero, J., Bustince, H., Barrenechea, E.: n-dimensional overlap functions. Fuzzy Sets and Systems **287**, 57 – 75 (2016).
6. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer, Dordrecht (2000)
7. Santos, H., Dimuro, G.P., Asmus, T., Lucca, G., Bueno, E., Bedregal, B., Bustince, H.: General grouping functions. In: IPMU 2020, Com. in Comput. and Inf. Science, vol. 1238, pp. 481–495. Springer, Cham (2020).