A note to pseudo-n-uninorms

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Abstract. The paper presents an approach to characterize all pseudo-n-uninorms with continuous underlying functions, i.e., non-commutative counterparts of n-uninorms. This approach differs significantly from the approach of characterizing n-uninorms via z-ordinal sum, which is no longer suitable for general pseudo-n-uninorms. The size of n is reduced inductively through the set of one-sided annihilators of a pseudo-n-uninorm. After this reduction, it is shown that each such reduced pseudo-n-uninorm is then only an ordinal sum of a pseudo-uninorm and a pseudo-n-uninorm, which can be further reduced.

Keywords: Pseudo-n-uninorm, n-uninorm, Annihilator, Pseudo-uninorm

The n-uninorms, which were proposed by Akkela in 2007 [1], unify the theory of both nullnorms and uninorms. The case of idempotent n-uninorms and n-uninorms with continuous underlying functions were completely characterized by Mesiarová-Zemánková in [4] and [5] respectively. For their characterization, she proposed a new construction method, which extends Clifford's ordinal sum, named z-ordinal sum.

Theorem 1 (z-ordinal sum, [4]). *Let A and B be two index sets such that $A \cap B = \emptyset$ and $C = A \cup B \neq \emptyset$. Let $(G_\alpha)_{\alpha \in C}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups and let the set C be partially ordered by the binary relation $\preceq$ such that $(C, \preceq)$ is a meet semi-lattice, Further suppose that each semigroup G_α for $\alpha \in A$ possesses an annihilator z_α, and for all $\alpha, \beta \in C$ such that α and β is incomparable there is $\alpha \vee \beta \in A$. Assume that for all $\alpha, \beta \in C$, $\alpha \neq \beta$, the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$. In the second case suppose that for all $\gamma \in C$ which is incomparable with $\alpha \vee \gamma = \beta \vee \gamma$ and for each $\gamma \in C$ with $\alpha \vee \prec \gamma \prec \alpha$ or $\alpha \vee \beta \prec \gamma \prec \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Further in the case that $\alpha \vee \beta \in A$ then $x_{\alpha,\beta} = z_{\alpha \vee \beta}$ is the annihilator of both G_β, G_α. And in the case that $\alpha \vee \beta = \alpha \in B$ then $x_{\alpha,\beta}$ is the annihilator of G_β and the neutral element of G_α.*

Put $X = \cup_{\alpha \in C} X_\alpha$ then $G = (X, *)$ is a semigroup if $*$ is defined as follows

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta \text{and } \alpha \vee \beta = \alpha \in B \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta \text{and } \alpha \vee \beta = \beta \in B \\ z_\gamma & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta \text{and } \alpha \vee \beta = \gamma \in A \end{cases}$$

Later on, these results led to the complete characterization of commutative associative aggregation functions continuous around the main diagonal on the unit interval in [6].

A similar characterization of non-commutative associative aggregation functions continuous around the main diagonal on the unit interval is still missing. Therefore, the main intention of this contribution is to take a step further for such characterization. Thus we are now focused on the characterization of pseudo-n-uninorms which form non-commutative extensions of n-uninorms.

Definition 1. *Let $P^n : [0,1]^2 \to [0,1]$ be a binary function then*

- *it posses an n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ if for each $i \in \{1, \ldots, n\}$ and $x \in [z_{i-1}, z_i]$*
$$P^n(x, e_i) = P^n(e_i, x) = x$$
holds, where $z_0 = 0$ and $z_n = 1$.
- *it is called pseudo-n-uninorm if it is associative, non-decreasing in both coordinates and possesses an n-neutral element. Commutative pseudo-n-uninorm is called n-uninorm.*

A pseudo-n-uninorm on the squares given by $[z_{i-1}, e_i]^2$ and $[e_i, z_i]^2$ for $i \in \{1, 2, ..., n\}$ reduces to the pseudo-t-norm respectively pseudo-t-conorm in the latter case. We will refer to them as the underlying functions of the pseudo-n-uninorm P^n. Under the assumption of continuity the underlying functions of P^n are commutative (i.e., t-norm, t-conorm) [3]. Starting from the most general case of pseudo-n-uninorm P^n with the continuous underlying function we will propose its complete characterization completely by distinguishing all possibilities and then inductively reducing the order of a pseudo-n-uninorm. The following Lemma appears to be useful.

Lemma 1. *Let $P^n : [0,1]^2 \to [0,1]$ be an n-pseudo-uninorm with n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ then for $i, j \in \{0, \ldots, n\}$, $i \leq j$ $P^n(z_i, z_j), P^n(z_j, z_i) \in \{z_i, z_{i+1}, ..., z_j\}$.*

The previous Lemma implies that $P^n(0, 1) = z_i$ and $P^n(1, 0) = z_j$ for some $i, j \in \{0, 1, ..., n\}$. Assuming $z_i \leq z_j$ we find out the following:

- P^n on $[0, z_i]^2$ is a pseudo-i-uninorm with $P^n(0, z_i) = P^n(z_i, 0) = z_i$.
- P^n on $[z_i, z_j]^2$ is a pseudo-$(j-i)$-uninorm with $P^n(z_i, z_j) = z_i$ and $P^n(z_j, z_i) = z_j$.
- P^n on $[z_j, 1]^2$ is a pseudo-$(n-j)$-uninorm with $P^n(z_j, 1) = P^n(1, z_j) = z_j$.
- P^n on $[z_i, 1] \times [0, z_i]$ is constantly equal to z_i.
- P^n on $[0, z_j] \times [z_j, 1]$ is constantly equal to z_j.

Therefore it only remains to examine values of P^n on the squares $[0, z_i]^2, [z_i, z_j]^2$, $[z_j, 1]^2$ and the rectangles $[0, z_i] \times [z_i, z_j], [z_j, 1] \times [z_j, z_i]$. We will at first start with the square $[z_i, z_j]^2$.

If there exists some $x \in [z_i, z_j]$ such that $P^n(x, z_i) = x$ then the following hold:

1. x is a left annihilator of P^n.
2. $x \in \{z_i, z_{i+1}, ..., z_j\}$.

Now on we will denote L the set of left annihilators of P^n. Note that in our setup such set L is non-empty since $\{z_i, z_j\} \subset L$. Consider that the previous inclusion holds properly then we may state the following Proposition.

Proposition 1. *Let $P^n : [0,1]^2 \to [0,1]$ be an n-pseudo-uninorm with continuous underlying functions such that $P^n(0,1) = z_i$ and $P^n(1,0) = z_j$ and L be the set of left annihilators of P^n then for each $z_k, z_l \in L$, $z_k < z_l$ such that there is no $z_m \in L$, $z_k < z_m < z_l$ the following hold.*

1. *P^n restricted to $[z_i, z_j]^2$ is isomorphic to a $(l-k)$-pseudo-uninorm $P^{(l-k)} : [0,1]^2 \to [0,1]$ with continuous underlying functions and two left annihilators namely $0, 1$.*
2. *$P^n(x,y) = z_k$, for each $(x,y) \in [z_k, z_l[\times [0, z_k]$.*
3. *$P^n(x,y) = z_l$, for each $(x,y) \in [z_k, z_l[\times [z_l, 1]$.*

Remark 1.
- Note that this proposition is stated in the form that characterizes the structure of a pseudo-n-uninorm P^n on the rectangles $[0, z_i] \times [z_i, z_j], [z_j, 1] \times [z_j, z_i]$ as well.
- Observe that unlike the case of n-uninorms with continuous which can be constructed via z-ordinal sum of semigroups (consisting of trivial semigroups and uninorms not necessarily proper), this is no longer true for the case of general pseudo-n-uninorm as it can be seen from this Proposition.
- The case when $z_j < z_i$ can be dealt analogously and is left to the reader due to a lack of space.

Previously we have reduced the general pseudo-uninorm P^n to respectively to pseudo-$i/(n-j)/(k-l)$-uninorm respectively. We have also characterized the values of general pseudo-n-uninorms outside squares, which lay along the main diagonal. To characterize pseudo-uninorms on these squares, we point out that all of them are isomorphic to pseudo-uninorm P^m on the unit interval for corresponding m. For such pseudo-uninorm P^m hold $P^m(0,1), P^m(1,0) \in \{0,1\}$ and $L \subset \{0,1\}$, assuming $P^m(0,1) \leq P^m(1,0)$ since the other inequality is just a dual case. Notice that this case covers also the commutative options of choice $z_i = z_j \in \{0,1\}$.

We can further reduce such pseudo-m-uninorm P^m as follows. Since $P^m(e_1, e_m)$ $= z_k$ and $P^m(e_m, e_1) = z_l$ for some $k, l \in \{1, 2, ..., m-1\}$. Because of the idempotency of both z_k, z_l we can define and ensure the existence of x_0, y_0 given by:

$$x_0 = \inf(x | P^n(x, \min(z_k, z_l))) = \min(z_k, z_l),$$

$$y_0 = \sup(y | P^n(y, \max(z_k, z_l))) = \max(z_k, z_l).$$

For all $x, y \in]x_0, \min(z_k, z_l)] \times [\max(z_k, z_l), y_0[$ it holds that $P^n(x,y) = z_k$ and $P^n(y,x) = z_l$. Moreover x_0, y_0 are idempotent points of P^n. Now we will divide the interval $[0,1]$ on 2 disjoint domains, namely the interior denoted by I and the exterior E. With $[0, x_0[,]y_0, 1] \subset E$ and $]x_0, y_0[\subset I$. x_0 belongs to E if and only if $P^n(x_0, \min(z_k, z_l)) = P^n(\min(z_k, z_l), x_0) = \min(z_k, z_l)$ and similarly y_0 belongs to E if and only if $P^n(y_0, \max(z_k, z_l)) = P^n(\max(z_k, z_l), y_0) = \max(z_k, z_l)$. In such case it holds that $P^n(i, e) = P^n(e, i) = e$, whenever $i \in I$ and $e \in E$. Notice that both I and E are closed on the operation P^n. Thus the pseudo-m-uninorm P^m can be constructed via Clifford's ordinal sum of two semigroups $G_1 = (E, P^m)$ and $G_2 = (I, P^m)$ with order $1 \prec 2$. Observe that (E, P^m) is a generalized pseudo-uninorm.

Since pseudo-uninorms with continuous underlying functions were characterized in [2], we will further focus only on semigroup $G_1 = (I, P^m)$. But in that case $z_k = P^m(0, 1)$ and $z_l = P^m$ are one sided annihilators of G_1 which is a pseudo-m-uninorm with continuous underlying functions on interval I. Such pseudo-m-uninorm can be then decomposed similarly as was described above.

We can proceed inductively until we reduce n to 1 and in that case, the pseudo-1-uninorm is only a pseudo-uninorm with continuous underlying functions. Now there remains an interesting open question. Whether there exists some non-commutative construction approach similar to the z-ordinal sum which is suitable for a similar characterization of pseudo-n-uninorms and thus for construction of other non-commutative associative functions.

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