The Compositional Nature of Weights

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Abstract. Commonly weights are viewed as monolithic quantities which can be estimated from data, elicited from experts or are known a priori. In this paper we question this assumption, positing that frequently weights reflect a series of underlying processes, each effectively contributing a component of the overall quantity making up the weight in the given aggregation or reasoning context. As such, we consider weights as a set of entangled components, defining them using the framework of compositional data, a field of statistics designed specifically to model components of a whole. We proceed by focusing on two well established processes–and thus weight components–in the field of data aggregation: the weighting of sources–underpinning linear averages, and the weighting of evidence–as underpinning ordered weighted averages. Using these two aggregation processes, we demonstrate how compositional geometry can be used to disentangle weights estimated from data according to the underlying processes and discuss the impact of doing so – including an improved capability for explaining both the weights and the weighting approaches.

Keywords: Aggregation · Compositional Geometry · Explainability · Entanglement.

1 Introduction

In the last two decades, extensive exploration into data-driven machine learning (ML) has prompted a reevaluation of the concept of artificial intelligence (AI), challenging our understanding of its attainability and proximity. Despite the complexity inherent in modern techniques, such as neural networks with trillions of weights, there often exists a reduced and more meaningful set of underlying processes that drive the task at hand. Articulating these is a crucial aspect of the increasingly important research focus on eXplainable AI (XAI). While numerous methods encode model information in a distributed and inaccessible manner, certain solutions are tailored to focus on explicit and centralized information representation via learning or design. As an illustration, a notable example of the former is the autoencoder (AE), along with its recent extensions like variational autoencoders (VAE) [10] for generative AI and the U-Net [12] (an AE with skip connections), while fuzzy systems are an example of the latter.

In a broad sense, substantial efforts in disciplines like statistics and computer science have been dedicated to the estimation of weights, frequently manifesting as embedding or latent spaces, or indeed explicitly as a set of weights of an associated aggregation operator. However, while contemporary data-driven methodologies have done well in estimating their overall *monolithic*[1] value, the task of disentangling these quantities, in particular into meaningful components, has proved challenging.

We focus on this problem of disentanglement specifically within the context of data aggregation. The latter boasts a rich history spanning decades, characterized by diverse underlying principles and frameworks. Ranging from straightforward linear strategies to more intricate non-linear methodologies like ordered weighted averages [17], fuzzy logic [18], fuzzy measures and integrals [14], and more recent works like pre-aggregation [5], among others, the landscape is expansive. While more sophisticated operators like fuzzy integrals capture "interactions" between sources (weights pertaining to higher order tuple couplings), the initial focus of our current article is on aggregation operators assuming independence among weights. Examples include the weighted average, which assigns weights to individual sources, and the OWA, which assigns weights to evidence based on rank ordering. These methods are applicable to a plethora of works across disciplines that either utilize or expand upon these concepts. Crucially, substantial efforts have also been invested into the combination of both linear and non-linear weighting strategies–and their associated weights– [6,11,15,16], as well as most recently, the Joint Weighted Averaged (JWA) [4], further discussed in Section 2.2.

In this paper, we recast the efforts of combining the above aggregation strategies as an example of combining effectively two underlying, atomic weighting strategies and weights into one overall aggregation strategy where the now monolithic weights represent an entanglement of the underlying strategies. We build on this conceptualisation to articulate and explore the reverse process, i.e. we posit that entangled weighting strategies and weights can, at least in some cases, be disentangled into their underlying component strategies and weights, offering a variety of benefits, including from an XAI perspective.

Specifically, we make the following contributions in this paper:

- We articulate the nature of traditionally monolithically viewed weights as disentanglable quantities from a compositional statistics perspective - a branch of statistics dedicated to modelling relative components which form a whole.
- We demonstrate disentanglement in an aggregation context using a synthetic example and leveraging the recently introduced joint weighted average, showing specifically how a set of monolithic weights, as could result from a weight

[1] We refer to monolithic as a condition in which the information in a set of weights or variables could be meaningfully separated into a set of sub-components.

estimation process, can be disentangled into distinct linear and order weight components.
- We discuss how the above example links to the more general real-world estimation of weights and lay out the challenges and future research of the proposed approach in terms of handling uncertainty in weight estimation and the complexity of high-dimensional weight estimation.

In Sect. 2 we give an overview of the framework of compositional statistics and the joint weighted average which leverages that framework. Sect. 3 discusses the meaning of weights and a general strategy for disentangling them. We then outline an algebraic approach and a geometric approach to disentanglement, as well as walk through numerical examples for each, in Sect. 4. In Sect. 5, we provide a discussion on the paper's findings. Finally, we outline future work and conclude in Sect. 6.

2 Background

While aggregation operators such as mentioned above are well known to the audience and are not further reviewed here, we proceed to give an overview of less common, yet core concepts leveraged throughout this paper, namely the framework of compositional data and statistics, followed by an overview of a recently introduced aggregation operator leveraging that framework–the joint weighted average.

2.1 Compositional Data and Statistics

A composition is a set of components that are constrained to sum to a constant (See [1,3,13] for tutorials). There are many compositions commonly used in research, such as proportions and probabilities. The relative proportion of outcomes from a random process (or their probability of occurrence) are compositions because they are constrained to sum to 1. If a random process leads to a larger observed proportion of one outcome, the observed proportion of the other must reduce. That is, each component part represents *relative* information, and therefore, cannot be analyzed separately, but must be analyzed as a part of a set. It is worth noting that zeros pose some unique problems for compositional operations. There are several strategies for handling zeros which are discussed in [13], but we do not deal with any zeros in the compositions used in this paper.

A collection of components with a fixed sum live on a mathematical structure called a simplex (Figure 1 displays a 3-dimensional simplex). The simplex has its own geometry, typically referred to as compositional geometry. Within the simplex, operations such as addition and multiplication take on a different meaning. Two compositions can be added together, but their sum results in perturbing the relative value of each part of the composition. Compositions with K parts are represented as vectors $\mathbf{x} = \{x_1, x_2, \ldots, x_K\}$ and $\mathbf{y} = \{y_1, y_2, \ldots, y_K\}$, and perturbation of composition $\mathbf{x}$ with composition $\mathbf{y}$ is given by

$$\mathbf{x} \oplus \mathbf{y} = \{\frac{x_1 * y_1}{\sum_{k=1}^{K} x_k * y_k}, \frac{x_2 * y_2}{\sum_{k=1}^{K} x_k * y_k}, \ldots, \frac{x_K * y_K}{\sum_{k=1}^{K} x_k * y_k}\}.$$

Perturbation shifts value from one part of the composition to another relative to the magnitude of the product between individual components. The composition $\mathbf{x}$ with $x_k = 1/K$ for all k is the additive identity in compositional geometry (adding this composition to any composition $\mathbf{y}$ yields $\mathbf{y}$).

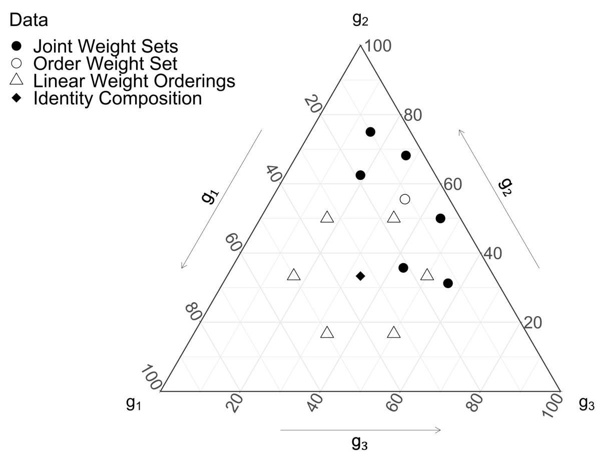


Fig. 1. Ternary diagram of example weights and identity composition

Due to the above definition of perturbation, multiplying a composition by a scalar results in raising each part of the composition to the power of the scalar. Given composition $\mathbf{x}$ and scalar a, the product is given by

$$\mathbf{x} * a = \{\frac{x_1^a}{\sum_{k=1}^{K} x_k^a}, \frac{x_2^a}{\sum_{k=1}^{K} x_k^a}, \ldots, \frac{x_K^a}{\sum_{k=1}^{K} x_k^a}\}.$$

Raising the parts to a power also moves the value associated with one part to another, with the shift based on the magnitude of each part. The multiplicative identity is the scalar $a = 1$ or the uniform composition where $x_k = 1/K, \forall k$.

To facilitate interpretation and statistical analysis, researchers project compositions onto a linear space, commonly with one fewer dimension [1,3,13]. There are several projections that are commonly used (see [1, 3, 13] for a review), we will use the Additive Log Ratio (ALR) transform as one of the most accessible isomorphic transforms in common use. The transform compares each component part against the K^{th} part using the following function:

$$\lambda_k = alr(x_k) = \log\left(x_k/x_K\right). \quad (1)$$

For each $k < K$, λ_k is a continuous variable on the real line, projecting the composition onto an $\mathbb{R}^{K-1}$ Euclidean space. Additionally, data in the Euclidean space can be projected back into the simplex with the following inverse function, making the ALR transform isomorphic:

$$x_k = alr^{-1}(\lambda_k) = \exp(\lambda_k) \Big/ \left(1 + \sum_{j=1}^{K-1} \exp(\lambda_j)\right). \tag{2}$$

$$x_K = 1 \Big/ \left(1 + \sum_{j=1}^{K-1} \exp(\lambda_j)\right). \tag{3}$$

2.2 Joint Weighted Average

The Joint Weighted Average (JWA) [4] is an aggregation operator that leverages compositional statistics to systematically combine two weighting strategies:

1. *The weighting of sources* based on their intrinsic value, such as the level of experience of a human expert, or the quality of a given sensor. On its own, the weighting of sources is most simply reflected by the *weighted arithmetic mean*, affording a linear aggregation of the information. It is attractive from an explainability point of view and particularly popular in the social sciences as a mechanism to explaining the importance of individual sources in respect to the scale of their contribution to a whole [7].
2. *The weighting of evidence* operationalized by the Ordered Weighted Average (OWA) [17], focuses not on the sources, but on the evidence *arising from* a given source. As a linear order statistic, the OWA leverages the order in respect to the scale or size of the evidence as a means to attribute a fixed vector of weights, commonly established a priori. By dynamically mapping the weights to the evidence as this arises, the operator achieves a non-linear aggregation, making it popular across engineering, computer and the physical sciences as a means to minimize error during automated aggregation.

While a number of authors previously explored combining the specific weighting strategies above [6, 11, 15, 16], the critical contribution of the JWA [4] is the systematic integration of both strategies and their weights, into one joint strategy–and weight–through the aforementioned framework of compositional geometry. Given K sources, and $\mathbf{w} = \{w_1, \ldots w_n\}$, and $\mathbf{v} = \{v_1, \ldots v_n\}$ sets of convex source and order weights respectively as introduced above, note that both $\mathbf{v}$ and $\mathbf{w}$ are compositions because their components are non-negative and sum to 1. Consider the evidence $\boldsymbol{x} = \{x_1, \ldots x_K\}$ generated by the K sources. For the JWA, both the evidence and the linear, source-specific weights are reordered by the same permutation function $\pi()$, such that $x_{\pi(1)} \geq x_{\pi(2)} \geq \ldots \geq x_{\pi(K)}$.

To combine the two sets of weights, the JWA applies the *perturbation* operation $\oplus$ (see Section 2.1), and the resulting *joint* weights are applied to the ordered evidence to generate the joint weighted average as follows:

$$JWA(\boldsymbol{x}) = (\mathbf{w}_\pi \oplus \mathbf{v})\boldsymbol{x}_\pi^T. \tag{4}$$

3 Disentangling Weights – Why and How

3.1 The Meaning of Weights

Similar to a regression problem, weights can be learned (or estimated) from investigating a data set that contains evidence from the information sources and their aggregates. If we pool the entire data set together to estimate the weights, we treat them as though they are uniformly applied by one specific process (or theory) across the dataset, i.e., the weights are monolithic. There are many statistical advantages to such pooling, however, these are only advantages if our theorized weighting procedure is uniformly applicable across the dataset.

In this paper, we argue that such traditional monolithic weights frequently hide underlying processes and effectively sub-weights which are tied to properties of the information being combined. While using monolithic weighting can lead to good overall estimation performance, it risks over-generalising, and perhaps even more crucially, it makes the resulting aggregation opaque as the disentangled underlying processes harbour the information critical for explanations.

For example, the behavioral sciences are interested in aggregating judgments from multiple experts, say medical doctors. We assign a weight to each expert source that is defined by their medical degree, past judgement accuracy, or years of experience. This weight represents the worth of that doctor's judgment in our aggregate. However, uniformly applying this weight cannot consider that the dose prescribed by an experienced doctor is above the safe limits in a given case. We may want to combine both strategies of considering expertise levels and making sure of safety margins - and both strategies can be combined to different degrees across multiple aggregates in a dataset, making the meaning of the weights change across context.

3.2 A General Approach to Disentanglement

As outlined above, the estimation of weights historically assumed a singular weighting process that is applied across all aggregates in a dataset. As a general approach to disentanglement, the researcher needs a theory that allows them to understand (or separate) the various contexts in which the weighting procedure may change. In essence, disentanglement first requires separating the data out so that only data points that apply the same weighing strategy are pooled.

We propose that a theorized set of weights can be represented as the entanglement of many types of weighting strategies, leading to a general representation of the aggregate based on a perturbation of many operating weighting strategies such as weight sets $\mathbf{w_1}$, $\mathbf{w_i}$, and $\mathbf{w_n}$ and error e as follows:

$$Agg(\boldsymbol{x}) = (\mathbf{w_1} \oplus \mathbf{w_i} \oplus \ldots \oplus \mathbf{w_n})\boldsymbol{x}^T + e. \tag{5}$$

The conditions under which the combined weighting strategies change determine unique entangled weight sets that can be used to solve for the underlying weighting strategies. If the number of estimated entangled weight sets is greater than the number of weights that need to be solved, then there is potential for unique disentanglement solution.

There is no one single answer to how this separation is achieved. However, the JWA specifies a specific process for weight entanglement, and with such a theory we can cleanly identify that permutations in the rank order of the evidence will lead to different entangled weights. This is because the joint weights are defined as a perturbation of the linear weights assigned to the source of the evidence with the order weights assigned to the ordered rank of the evidence. When the ordered rank of the evidence changes, then the entangled weights will change. Therefore, we can capture the behavior of the combination of linear and order weights by separately analyzing the weighting behavior for each observed permutation of the evidence. For this approach, compositional geometry is essential in solving for the disentanglement solution.

In the next section we demonstrate disentanglement using a synthetic example. We leverage the JWA, showing specifically how a set of monolithic weights, as could result from a weight estimation process, can be disentangled into distinct linear and order weight components.

Table 1. Weights for Numerical Examples (rounded to 2 digits)

	Source 1	Source 2	Source 3	
Linear Weights	$g(x_1) = 0.17$	$g(x_2) = 0.33$	$g(x_3) = 0.50$	
	Rank 1	Rank 2	Rank 3	Permutation
Order Weights	$g_1 = 0.11$	$g_2 = 0.56$	$g_3 = 0.33$	
Joint Weight Set j_1	0.05	0.50	0.45	1 2 3
Joint Weight Set j_2	0.05	0.68	0.27	1 3 2
Joint Weight Set j_3	0.13	0.31	0.56	2 1 3
Joint Weight Set j_4	0.10	0.75	0.15	2 3 1
Joint Weight Set j_5	0.21	0.36	0.43	3 1 2
Joint Weight Set j_6	0.19	0.62	0.19	3 2 1

4 Disentangling the Joint Weighted Average Using Compositional Methods

To demonstrate the disentanglement of weights using compositional methods, we work through an example using the joint weighted average (JWA) operator [4].

For real data, the first step is to estimate the sets of entangled weights from the dataset. For our example, we create the sets of entangled weights from a specific set of linear weights and a specific set of order weights. We are using 3 sources, so there are $3! = 6$ possible rank orders of the data in X. The weight sets extracted from the 6 orderings are shown in Table 1. We will show two distinct strategies for obtaining the original linear weight set and order weight set using only the entangled joint weight sets.

Let the joint weights be $\mathbf{j} = (\mathbf{w}_\pi \oplus \mathbf{v})$. Because $\mathbf{j}$ is a function of $\mathbf{w}_\pi$ and $\mathbf{v}$, the weights in $\mathbf{j}$ can change for each permutation of $\mathbf{w}_\pi$. Let $\mathbf{j}_p$ represent each unique set of weights that is possible for all permutations of evidence from K sources, such that $p = 1 \dots K!$. For completeness, we label the permutations of the linear weights with the same index, such that $\mathbf{j}_p = (\mathbf{w}_p \oplus \mathbf{v})$.

Compositional geometry provides the mathematical derivations required to solve for the unknown weights. There are several lemmas that can be derived from this basic setup, including the following.

Lemma 1. *The compositional sum of all K! permutations of a composition is equal to the additive identity composition.*

Proof. Let each permutation of the linear weights be the composition $\mathbf{w}_p = \{w_{p1}, w_{p2}, \dots, w_{pK}\}$. Let the compositional sum of all permutations be $\mathbf{w}_{.} = \mathbf{w}_1 \oplus ... \oplus \mathbf{w}_{K!}$. For $K = 3$, we have $3! = 6$ possible permutations as shown in the last column of Table 1. It is a general result that each rank order will appear in each component exactly $(K-1)!$ times. For the example of $K = 3$, we get each rank order in each component $2! = 2$ times. Adding all permutations together in compositional geometry requires taking the product of the elements in each component repeated $(K-1)!$ times given by,

$$x = \left(\prod_{k=1}^{K} (w_{1k}) \right)^{(K-1)!} . \tag{6}$$

Then we perform the closure operation to ensure that all components sum to one. The result is a composition with each component equal to

$$w_{.k} = \frac{x}{\sum_{k=1}^{K} x} = \frac{x}{K * x} = 1/K. \tag{7}$$

Therefore, $\mathbf{w}_{.}$ is equal to the uniform composition of $1/K$.

4.1 Algebraic Decomposition of Joint Weights

For any value of K, we can solve for the linear weights $\mathbf{w}$ and order weights $\mathbf{v}$ from all sets of joint weights $\mathbf{j}_p$ using the above lemma. Let I be the identity composition where each component is $1/K$. If we have all $K!$ joint weight sets, then the sum of the joint weights sets can be solved as the following:

$$\begin{aligned} \mathbf{j}_1 \oplus \mathbf{j}_2 ... \oplus \mathbf{j}_{K!} = (\mathbf{w}_1 \oplus \mathbf{v}) \oplus (\mathbf{w}_2 \oplus \mathbf{v}) \oplus \dots \oplus (\mathbf{w}_{K!} \oplus \mathbf{v}) = \\ (\mathbf{w}_1 \oplus \mathbf{w}_2 \oplus \dots \oplus \mathbf{w}_{K!}) \oplus (K!)\mathbf{v} = I \oplus (K!)\mathbf{v} = (K!)\mathbf{v}. \end{aligned}$$

We can therefore solve for the order weights as the following:

$$\mathbf{v} = \frac{\mathbf{j}_1 \oplus \mathbf{j}_2 ... \oplus \mathbf{j}_{K!}}{K!} \tag{8}$$

Finally, we can solve for any permutation of the linear weights:

$$\mathbf{w}_p = \mathbf{j}_p \oplus (-1)\mathbf{v} \tag{9}$$

Numerical Example of the Algebraic Approach The 6 joint weight sets shown in Table 1 represent the entangled weights. They are displayed in the ternary plot in Figure 1. We can use Equations 8 and 9 to recover the disentangled linear and order weights shown at the top of Table 1. First, applying Equation 8 gives

$$(\{0.19, 0.62, 0.19\} \oplus \{0.21, 0.36, 0.43\} \oplus \{0.13, 0.31, 0.56\} \oplus \{0.10, 0.75, 0.15\} \oplus \{0.05, 0.68, 0.27\} \oplus \{0.05, 0.50, 0.45\})/3! = \{0.11, 0.56, 0.33\}.$$

This result is equal to the order weight set in Table 1. Second, applying Equation 9 with the recovered order weight set gives

$$\{0.05, 0.50, 0.45\} \oplus (-1)\{0.11, 0.56, 0.33\} = \{0.17, 0.33, 0.50\}.$$

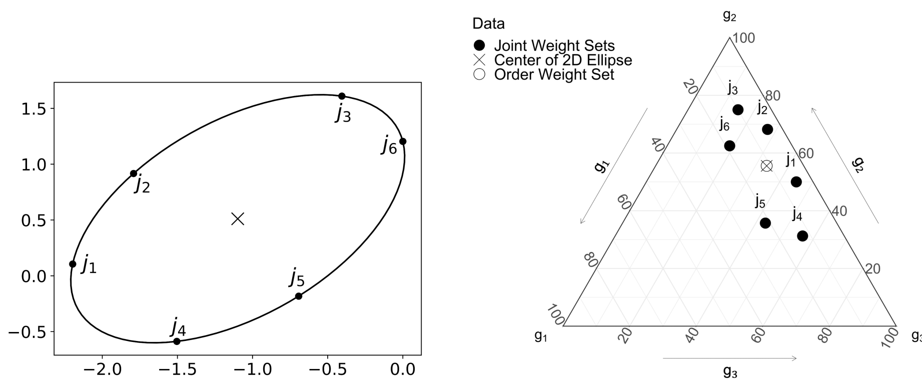
This result is equal to the linear weight set in Table 1.

4.2 Geometric Decomposition of Joint Weights

The algebraic decomposition of the joint weight sets is limited because Lemma 1 requires the sum of all $K!$ joint weight sets. As the number of sources K grows, solving for $K!$ sets of weights will become unwieldy. As a solution to this problem, we propose to leverage the geometric structure of the JWA model using compositional geometry as a more general approach to disentanglement that does not require all $K!$ joint weight sets.

As displayed in Figure 1, the joint weight sets form on the borders of an ellipse in the simplex, and the order weight set is located at the center of the ellipse. Equation 8 shows that the average of $K!$ points located at symmetrically spaced intervals around the ellipse is equal to the order weights. This average is uniquely defined as the center of the ellipse formed by the $K!$ joint weight sets. Due to the computational complexity of estimating ellipses in the simplex [9], we used Equation 1 to project this compositional ellipse into Euclidean space and obtained a $K-1$ dimensional euclidean ellipse. Therefore, we only need to obtain the minimal number of joint weight sets that will allow for accurate estimation of an ellipse in $K-1$ dimensions (i.e., the ALR projection of the compositional ellipse in Euclidean space). Then we can use the inverse ALR projection of the center of the ellipse (Equation 2) to recover the order weights and Equation (9) to recover the linear weights.

Numerical Example of the Geometric Approach Using the joint weight sets in Table 1, we projected these points in to a 2 dimensional Euclidean space using equation 1, see Figure 2a. We fit an ellipse to the points using a least squares minimization algorithm [8]. Fitting the ellipse gives us the 2D point that denotes the center of the ellipse, which can be seen on the figure. Note that for the sake of seeing all the joint weight sets in Euclidean space, we fit the ellipse to all 6 points, but any combination of 5 of the joint weight sets will yield the same ellipse since 5 points uniquely determine a 2D ellipse. Next, we project the ellipse center along with the other points back to the simplex using the inverse ALR function from Equation 2. Figure 2b displays the fitted ellipse center from Euclidean space in the simplex after projection with the inverse ALR. The ellipse center perfectly matches the order weights. With the joint weight sets and order weights in hand, we can then use Equation 9 to obtain all permutations of the linear weights as we did in the algebraic example.



(a) Ellipse in 2D Euclidean Space (b) Ternary plot of the data

Fig. 2. (a) Joint weight sets projected to 2D Euclidean Space via the ALR and fitted with an ellipse. (b) The center of the ellipse from 2D space projected to the simplex via the inverse ALR, displayed with the original order weight set and joint weight sets.

5 Discussion

Our synthetic example demonstrates how framing weights as compositions opens the door to their disentanglement–within the specific aggregation context discussed in this paper. Compositional geometry revealed the entanglement structure such that an ellipse fitting strategy could be employed to reduce the number of joint weight sets needed to recover the unentangled weights. In our example with K=3 sources, the algebraic solution required 3!=6 weight sets, but compositional geometry reduced that number to 5 weight sets (the number of points needed to uniquely define an ellipse in 2D Euclidean space, assuming no noise). Increasing K by 1 would require 4!=24 weight sets to recover the unentangled weights, yet only 9 joint weight sets are needed to fit a 3D ellipsoid in 3D Euclidean space [2]. For these two cases, the geometric approach reduces the number of weight sets needed to solve for the disentangled weights compared to the algebraic approach. Further research is needed to establish the minimum number of points needed for higher dimensional ellipsoids.

Different disentanglement strategies may become more advantageous as the number of weights increases. Some disentanglement strategies may be unidentifiable. Disentangling weights with different underlying weighting structures will likely involve uniquely tailored approaches, but viewing the weights as compositions can facilitate novel and efficient solutions, should they exist.

6 Conclusions and Future Work

In this paper, we began with an explanation on how weights that are typically viewed monolithically can be viewed as disentanglable quantities through the lens of compositional statistics. After discussing why these disentangled weights have importance across a wide variety of aggregation use cases, we walk through a synthetic example that demonstrates the disentanglement process, explaining how to disentangle weights leveraging compositional geometry.

In practice, monolithic weights are not typically estimated perfectly, thus introducing uncertainty into the process. It is beyond the scope of this paper, but future work will focus on how uncertainty associated with the estimated points results in uncertainty in the estimated ellipse. As part of a forthcoming journal paper, we expect to showcase how the proposed decomposition can not only improve explainability, but also support more efficient estimation of weights in practice, as well as afford the capability to estimate the quality of an extant estimation of weights.

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