Characterizing fully true attribute implications*

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Abstract. The theory of attribute implications is a fundamental line of research in formal concept analysis. The fully true attribute implications are the most significant attributes in the Boolean framework, and also play a fundamental role in the fuzzy one. This paper shows that the residuated concept lattice framework satisfies natural extensions of the equivalences among the definition of validity and the characterizations using the extents of the subsets of attributes. Moreover, it introduces sufficient conditions in order to obtain analogous equivalences in the multi-adjoint framework.

Keywords: Concept lattices · Fuzzy sets · Attribute implications.

1 Introduction

Formal Concept Analysis, introduced by Wille in the 1980s, is a useful mathematical tool for data analysis, which has become an important and appealing research topic. The computation of relationships between the variables involved in a given dataset, called attribute implications, is a key task in this theory, since obtaining a set of rules capable of modeling a given dataset is a fundamental goal to know the behaviour of the system to be studied, for instance, stock market prediction, disease diagnosis and census data analysis, among others.

Attribute implications have been widely studied from a Boolean [8,30] and fuzzy [4,5,20,22,23] perspective. Ganter and Wille introduced in [19] the Boolean definition of validity of an attribute implication, and proved that it is directly related to the inclusion of the extents of the subset of attributes envolved in the implication. The fuzzy generalizations of attribute implications allows to provide a degree of validity between the bottom and top of the considered complete lattice to each implication. However, the fully true implications, that is, attribute

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implications whose degree of validity is equal to 1, continue playing a fundamental role in the fuzzy case, as it was remarked in [6]. For example, it is possible to translate an implication to a fully true implication. This translation is also related to the f-index of inclusion [25,26].

Fuzzy formal concept analysis is a mathematical tool, based on lattice theory, to extract knowledge from relational datasets, which are formally translated into a set of objects, a set of attributes and a relationship between them, containing imperfect, incomplete and vague information [2,3,7,28,27]. This paper focuses on the study of attribute implications in residuated concept lattices [4] and multi-adjoint concept lattices [9,12,24]. Specifically, this paper will show that the fully true attribute implications in the residuated framework satisfy similar equivalences to the Boolean one, and what sufficient conditions are required in the multi-adjoint framework. The obtained results also show the really important properties needed in the residuated framework in order to obtain the equivalence among the definition of validity and the characterization using the extents of the subsets of attributes.

2 Preliminaries

This section will start recalling the basic notions of the residuated concept lattice framework [3].

Definition 1. *Given a complete lattice $(L, \leq_L)$ with bottom and top elements, $\bot$ and $\top$ respectively, a non-empty set X of objects, a non-empty set Y of attributes, and a L-fuzzy relation $I\colon X \times Y \to L$ which associates any element $(x, y) \in X \times Y$ with a truth value $I(x, y)$, we have that:*

- *A* complete residuated lattice *is the tuple $(L, \leq_L, *, \to)$, where $(L, *, \top)$ is a commutative monoid and $(*, \to)$ is a* residuated pair*, that is, the following equivalence holds, for all $x, y, z \in L$:*

$$x * y \leq_L z \qquad \text{if and only if} \qquad x \leq_L y \to z$$

- *A* context *is the tuple (X, Y, I).*
- *The* concept-forming operators[1] *${}^{\Uparrow}\colon L^X \to L^Y$ and ${}^{\Downarrow}\colon L^Y \to L^X$ are defined, for all $G \in L^X$, $F \in L^Y$, $x \in X$, $y \in Y$, as:*

$$G^{\Uparrow}(y) = \bigwedge_{x\in X}(G(x) \to I(x, y))$$
$$F^{\Downarrow}(x) = \bigwedge_{y\in Y}(F(y) \to I(x, y))$$

- *A* concept *is a pair $\langle G, F\rangle$ satisfying that $G^{\Uparrow} = F$, $F^{\Downarrow} = G$, for all $G \in L^X$ and $F \in L^Y$. The fuzzy subsets G and F are usually known as* the extent and intent of the concept*, respectively.*

[1] Given two sets R and S, the set R^S denotes the set of mappings $f\colon S \to R$.

- *The* residuated concept lattice*, denoted as* $(\mathcal{B}, \leq)$*, associated with the residuated complete lattice and the context, is the set:*

$$\mathcal{B} = \{\langle G, F\rangle \mid G \in L^X, F \in L^Y, G^{\Uparrow} = F, F^{\Downarrow} = G\}$$

together with the ordering $\leq$*, defined as* $\langle G_1, F_1\rangle \leq \langle G_2, F_2\rangle$ *if and only if* $G_1 \leq_L G_2$ *(or equivalently,* $F_2 \leq_L F_1$*). The sets of extents and intents of the concepts of* $(\mathcal{B}, \leq)$ *will be denoted as Ext*$(\mathcal{B})$ *and Int*$(\mathcal{B})$*, respectively.*

Next, the basic notions of the multi-adjoint concept lattice framework are recalled. This setting arose as a fuzzy generalization of FCA [28], which has given rise to relevant advances in different areas, such as attribute reduction [1,14], size reduction [11,13] and attribute implications [9,12,24].

Definition 2. *Given two complete lattices* $(L_1, \preceq_1)$*,* $(L_2, \preceq_2)$*, a poset* $(P, \leq_P)$*, a non-empty set* X *of objects, a non-empty set* Y *of attributes, and a* P*-fuzzy relation* $I\colon X \times Y \to P$*, we have that*

- *The tuple* $(L_1, L_2, P, \preceq_1, \preceq_2, \leq_P, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ *is a* multi-adjoint frame*, where* $(\&_i, \swarrow^i, \nwarrow_i)$ *are adjoint triples [15] with respect to* $(L_1, \preceq_1)$*,* $(L_2, \preceq_2)$ *and* $(P, \leq_P)$*, for all* $i \in \{1, \dots, n\}$*, that is, the following double equivalence holds, for all* $x \in L_1$*,* $y \in L_2$*,* $z \in P$*:*

$$x \preceq_1 z \swarrow^i y \quad \text{if and only if} \quad x \&_i y \leq_P z \quad \text{if and only if} \quad y \preceq_2 z \nwarrow_i x$$

- *A* context *is the tuple* (X, Y, I, σ)*, where* $\sigma\colon X \times Y \to \{1, \dots, n\}$ *is a mapping which associates any element in* $X \times Y$ *with a particular adjoint triple in the multi-adjoint frame.*
- *The* concept-forming operators $^{\uparrow_\sigma}\colon L_2^X \to L_1^Y$*,* $^{\downarrow^\sigma}\colon L_1^Y \to L_2^X$ *are defined, for all* $g \in L_2^X$*,* $f \in L_1^Y$*,* $x \in X$*,* $y \in Y$*, as:*

$$g^{\uparrow_\sigma}(y) = \inf\{I(x', y) \swarrow^{\sigma(x',y)} g(x') \mid x' \in X\}$$
$$f^{\downarrow^\sigma}(x) = \inf\{I(x, y') \nwarrow_{\sigma(x,y')} f(y') \mid y' \in Y\}$$

- *A* concept *is a pair* $\langle g, f\rangle$ *satisfying that* $g^{\uparrow_\sigma} = f$*,* $f^{\downarrow^\sigma} = g$*, for all* $g \in L_2^X$ *and* $f \in L_1^Y$*. The fuzzy subsets* g *and* f *are usually known as* the extent and intent of the concept*, respectively.*
- *The* multi-adjoint concept lattice*, denoted as* $(\mathcal{M}, \preceq)$ *and associated with the multi-adjoint frame and the context, is the set:*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^X, f \in L_1^Y, g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

together with the ordering $\preceq$*, defined as* $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *if and only if* $g_1 \preceq_2 g_2$ *(or equivalently,* $f_2 \preceq_1 f_1$*). The sets of extents and intents of the concepts of* $(\mathcal{M}, \preceq)$ *will be denoted as Ext*$(\mathcal{M})$ *and Int*$(\mathcal{M})$*, respectively.*

From now on, in order to simplify the notation, we write $g^\uparrow$ and $f^\downarrow$ instead of $g^{\uparrow_\sigma}$ and $f^{\downarrow^\sigma}$, respectively. It is important to mention that the pair of concepts-forming operators $(^\uparrow, ^\downarrow)$ forms an antitone Galois connection [28], whose properties play a key role in the proofs.

Definition 3. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets. We say that the pair $(^\uparrow, ^\downarrow)$ of mappings $^\downarrow \colon P_1 \to P_2$, $^\uparrow \colon P_2 \to P_1$ is an* antitone Galois connection *between P_1 and P_2 if the following properties are satisfied:*

(1) $^\uparrow$ and $^\downarrow$ are order-reversing.
(2) $x \leq_1 x^{\downarrow\uparrow}$ and $y \leq_2 y^{\uparrow\downarrow}$
(3) $x^\downarrow = x^{\downarrow\uparrow\downarrow}$ and $y^\uparrow = y^{\uparrow\downarrow\uparrow}$

for all $x \in P_1$ and $y \in P_2$.

Now, we present the notion of forcing-implication, which play an important role in different results obtained in this paper.

Definition 4. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets such that $\top_1$ is the top element in $(P_1, \leq_1)$. We say that the mapping $\swarrow \colon P_2 \times P_2 \to P_1$ is a* forcing-implication *if it is order-preserving in the left argument, order-reversing in the right argument and the following equivalence holds, for all $y, z \in P_2$:*

$$z \swarrow y = \top_1 \quad \textit{if and only if} \quad y \leq_2 z$$

Finally, we include some properties associated with adjoint triples in order to make the paper self-contained.

Proposition 1 ([10]). *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$. If $P_2 \subseteq P_3$ and P_1 has a maximum $\top_1$, the following equivalence holds:*

$$\swarrow \textit{ is a forcing-implication} \quad \textit{if and only if} \quad \top_1 \& y = y, \textit{ for all } y \in P_2.$$

Proposition 2 ([17]). *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the poset $(P, \leq)$. Then, for all $x, y, z \in P$, the following equivalence holds:*

$$(z \swarrow y) \nwarrow x = (z \nwarrow x) \swarrow y \quad \textit{if and only if} \quad \& \textit{ is associative.}$$

3 Fully true attribute implications

We recall the philosophy underlying the notion of validity of an attribute implication [19]. Given two subsets of attributes A and B of a context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$, the Boolean definition of attribute implications given by Ganter and Wille in [19] established that the implication $A \to B$ is valid in the context, if every intent of a single object respect the implication. They proved that this definition is equivalent to the extent of A is included in the extent of B. Specifically, they obtain the following result.

Proposition 3 ([21]). *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, and $A, B \subseteq \mathcal{P}$, we have the following equivalence:*

$$A \to B \textit{ is valid in } \mathcal{C} \quad \textit{if and only if} \quad A^\downarrow \subseteq B^\downarrow \quad \textit{if and only if} \quad B \subseteq A^{\downarrow\uparrow}$$

where $^\downarrow$ and $^\uparrow$ are the derivation operators [19].

Next, different notions to compute the validity of attribute implications will be shown, together with useful properties for obtaining similar results to Proposition 3, concerning fully true attribute implications, that is, attribute implications whose degree of validity is equal to 1.

3.1 Validity of residuated attribute implications

Residuated attribute implications have been widely studied [6,18,29]. Specifically, [6] addresses the study of residuated attribute implications and the issues traditionally investigated for attribute implications such as validity, entailment, redundancy and bases of attribute implications associated with closure structures. Taking into consideration [6], this section starts including the syntactic definition of residuated attribute implication together with the notions related to its semantic interpretation. From now on, a complete residuated lattice $(L, \leq_L, *, \rightarrow)$ and a context $\langle X, Y, I\rangle$ will be fixed.

Definition 5. *Given two fuzzy subsets of attributes $A, B \in L^Y$, we say that the expression $A \Rightarrow B$ is a* residuated attribute implication *over Y.*

The degree of validity of a residuated attribute implication allows to provide a semantic interpretation to the above definition. After introducing the usual notion of fuzzy inclusion, which is a generalization of the inclusion relation of classical sets, we recall the degree of validity of a residuated attribute implication both in a fuzzy subset of attributes and in a family of fuzzy subsets of attributes.

Definition 6. *Let $\mathcal{H}$ be a family of fuzzy subsets of attributes in L^Y and $A, B \in L^Y$ be two fuzzy subsets of attributes.*

- *$S(A, B)$ is the degree in which A is included in B, defined as:*

$$S(A, B) = \bigwedge_{y\in Y} (A(y) \rightarrow B(y))$$

- *$\|A \Rightarrow B\|_H$ is* the degree in which $A \Rightarrow B$ is valid in $H \in L^Y$, *defined as:*

$$\|A \Rightarrow B\|_H = S(A, H) \rightarrow S(B, H).$$

- *$\|A \Rightarrow B\|_{\mathcal{H}}$ is* the degree in which $A \Rightarrow B$ is valid in $\mathcal{H}$, *defined as:*

$$\|A \Rightarrow B\|_{\mathcal{H}} = \bigwedge_{H\in\mathcal{H}} \|A \Rightarrow B\|_H$$

The definition of validity of a residuated attribute implication was also extended to a given context in [6] as follows.

Definition 7 ([6]). *Given two fuzzy subsets of attributes $A, B \in L^Y$, the degree in which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ is defined as:*

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\{I_x \mid x\in X\}}$$

where I_x is the x-th row of the table associated with the relation I, that is, $I_x(y) = I(x, y)$, for all $y \in Y$.

Notice that, the validity of a residuated attribute implication in a given context is interpreted as the degree of validity in the whole set of rows of its relational table. In addition, this degree of validity can be characterized by the degree of validity on the whole set of intents of the residuated concept lattice, which coincides with the degree of inclusion of the consequent set in the closure of the antecedent set, as the following theorem shows.

Theorem 1 ([6]). *Given two fuzzy subsets of attributes $A, B \in L^Y$, we obtain that:*

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{Int(X,Y,I)} = S(B, A^{\Downarrow\Uparrow}).$$

where $Int(X, Y, I)$ is the set of intents of the concept lattice $(\mathcal{B}, \leq)$.

In the fuzzy framework, practical considerations often lead to work with fully true attribute implications. For this reason, we will now focus our study on presenting different properties helpful to obtain a characterization of fully true residuated attribute implications.

First of all, we include a property which relates the degree of validity of an attribute implication $A \Rightarrow B$ in the given context to the degree of inclusion $S(A^{\Downarrow}, B^{\Downarrow})$, whose proof is included in the proof of the previous theorem in [6].

Proposition 4. *Given two fuzzy subsets of attributes $A, B \in L^Y$, we have that:*

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = S(A^{\Downarrow}, B^{\Downarrow})$$

Finally, we provide the equivalences required for obtaining fully true residuated attribute implications, that is, the fuzzy extension of Proposition 3. Its proof straightforwardly follows from Proposition 4 and the well-known property that $A \leq_L B$ if and only if $S(A, B) = 1$.

Proposition 5. *Given two fuzzy subsets of attributes $A, B \in L^Y$, we have that:*

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = 1 \quad \textit{if and only if} \quad A^{\Downarrow} \leq_L B^{\Downarrow} \quad \textit{if and only if} \quad B \leq_L A^{\Downarrow\Uparrow}$$

From a practical standpoint, this property plays a fundamental role to obtain fully true attribute implications from a given context, since it is not needed to compute the degree of validity through the rows of the incidence relation. This fact is illustrated in the following example.

Example 1. Let $([0, 1], \leq_{[0,1]}, *_L, \rightarrow_L)$ be a complete residuated lattice where $(*_L, \rightarrow_L)$ is the Łukasiewicz residuated pair and (X, Y, I) be a context whose set of objects is $X = \{x_1, x_2, x_3\}$, the set of attributes is $Y = \{y_1, y_2, y_3, y_4\}$, and the relation $I \colon X \times Y \rightarrow [0, 1]$ is given by Table 1.

Now, we will consider the fuzzy subsets of attributes $A = \{1/y_1, 0.5/y_3\}$ and $B = \{0.5/y_2, 0.5, y_3, 0.5/y_4\}$ and we will check that $A \Rightarrow B$ is a fully true residuated attribute implication from Proposition 5. Applying the definition of the operator $^{\Downarrow}$, we have that:

$$\begin{aligned} A^{\Downarrow}(x_1) &= \inf\{(A(y) \rightarrow_L I(x_1, y)) \mid y \in Y\} = 1 \\ A^{\Downarrow}(x_2) &= \inf\{(A(y) \rightarrow_L I(x_2, y)) \mid y \in Y\} = 1 \\ A^{\Downarrow}(x_3) &= \inf\{(A(y) \rightarrow_L I(x_3, y)) \mid y \in Y\} = 0.9 \end{aligned}$$

I	y_1	y_2	y_3	y_4
x_1	1	0.9	0.8	1
x_2	1	0.7	0.8	1
x_3	0.9	0.5	0.8	1

$$x *_{\text{L}} y \;= \max\{0, x + y - 1\}$$
$$x \to_{\text{L}} y = \min\{1, 1 - x + y\}$$

Table 1. Relation I and the Łukasiewicz residuated pair used in Example 1.

Then, we obtain that $A^{\Downarrow} = \{1/x_1, 1/x_2, 0.9/x_3\}$. Carrying out similar analogous computations by using the operator $^{\Uparrow}$, we have that $B^{\Downarrow} = \{1/x_1, 1/x_2, 1/x_3\}$. Taking into account that the hypothesis of Proposition 5 are satisfied, that is:

$$A^{\Downarrow} = \{1/x_1, 1/x_2, 0.9/x_3\} \subseteq \{1/x_1, 1/x_2, 1/x_3\} = B^{\Downarrow}$$

or equivalently,

$$B = \{0.5/y_2, 0.5, y_3, 0.5/y_4\} \subseteq \{1/y_1, 0.6/y_2, 0.8/y_3, 1/y_4\} = A^{\Downarrow\Uparrow}$$

we can ensure that the residuated attribute implication $A \Rightarrow B$ is fully true in the given context. □

3.2 Validity of multi-adjoint attribute implications

Multi-adjoint attribute implications have been studied for instance in [9,12,16,24]. In [9] a novel definition of validity for multi-adjoint attribute implications on a context was proposed in order to consider the whole set of intents, which is not equivalent to the one given in the residuated framework. Following the same philosophy that the previous section and taking into account [9], we will introduce the syntactic definition of multi-adjoint attribute implication as well as the notions related to its semantic interpretation. Henceforth, a multi-adjoint frame $(L_1, L_2, P, \preceq_1, \preceq_2, \leq_P, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ and a context (X, Y, I, σ) will be fixed.

Given $f_1, f_2 \in L^Y$, we say that the expression $f_2 \Leftarrow f_1$ is a *multi-adjoint attribute implication* over Y. The following definition provides the semantic interpretation of these implications.

Definition 8. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$, $\mathcal{F}$ be a family of fuzzy subsets of attributes in L_1^Y, $f_1, f_2, f_3 \in L_1^Y$ be three fuzzy subsets of attributes and $g_1, g_2 \in L_2^X$ be two subsets of objects.*

- $S^1(f_1, f_2)$ *is* the degree in which f_1 is included in f_2, *defined as:*

$$S^1(f_1, f_2) = \bigwedge_{y \in Y} (f_2(y) \nwarrow f_1(y))$$

- $S^2(g_1, g_2)$ *is* the degree in which g_1 is included in g_2, *defined as:*

$$S^2(g_1, g_2) = \bigwedge_{x \in X} (g_2(x) \swarrow g_1(x))$$

- $\|f_2 \Leftarrow f_1\|_{f_3}$ *is* the degree in which $f_2 \Leftarrow f_1$ is valid in f_3*, defined as:*

$$\|f_2 \Leftarrow f_1\|_{f_3} = S^1(f_2, f_3) \swarrow S^1(f_1, f_3)$$

- $\|f_2 \Leftarrow f_1\|_{\mathcal{F}}$ *is* the degree in which $f_2 \Leftarrow f_1$ is valid in $\mathcal{F}$*, defined as:*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{F}} = \bigwedge_{f \in \mathcal{F}} \|f_2 \Leftarrow f_1\|_f$$

The notion of validity of a multi-adjoint attribute implication was also extended to a multi-adjoint concept lattice [9], as it is shown below.

Definition 9. *Given two subsets of attributes $f_1, f_2 \in L_1^Y$, the degree in which the multi-adjoint attribute implication $f_2 \Leftarrow f_1$ is valid in the multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ is defined as:*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \|f_2 \Leftarrow f_1\|_{Int(\mathcal{M})}$$

where $Int(\mathcal{M})$ is the set of intents of the concepts of $(\mathcal{M}, \preceq)$.

Unlike to the residuated framework, the definition of validity on the whole set of intents is not equivalent to the definition on the whole set of rows of the relation table, as it was shown in [9, Example 26]. Furthermore, in the general multi-adjoint framework, only one inequality in the equality displayed in Proposition 4 can be ensured. Specifically, the degree of inclusion $S^2(f_1^{\downarrow}, f_2^{\downarrow})$ is not equal to the degree of validity $\|f_2 \Leftarrow f_1\|_{\mathcal{M}}$ based on the whole set of intents. We can only assert that the degree of inclusion $S^2(f_1^{\downarrow}, f_2^{\downarrow})$ is an upper bound of $\|f_2 \Leftarrow f_1\|_{\mathcal{M}}$ in the multi-adjoint framework, as the following theorem shows.

Theorem 2. *Let $(L_1, \preceq_1)$ be a complete lattice satisfying the descendent chain condition, $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $x \& \top_1 = x$, for all $x \in L_1$, and $f_1, f_2 \in L_1^Y$ be two fuzzy subsets of attributes. Then:*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}} \preceq_1 S^2(f_1^{\downarrow}, f_2^{\downarrow})$$

Following the same sequence of results as in the previous section, we will now devote to the study to analyze different properties useful to characterize fully true multi-adjoint attribute implications. The first result provides a sufficient condition to ensure the usual equivalence given by the properties of a Galois connection, but now when the fuzzy inclusions S^2 and S^1 are considered.

Proposition 6. *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$ such that $\&$ is associative and two subsets of attributes $f_1, f_2 \in L_1^Y$, then we have that:*

$$S^2(f_1^{\downarrow}, f_2^{\downarrow}) = S^1(f_2, f_1^{\downarrow\uparrow})$$

Next technical property is necessary before presenting the results associated with fully true multi-adjoint attribute implications. Notice that, although it seems trivial, an extra property is required on the general adjoint implications to obtain it.

Proposition 7. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\swarrow$ is a forcing-implication. Given two subsets of objects $g_1, g_2 \in L_2^X$, then we have that:*

$$g_1 \preceq_1 g_2 \quad \textit{if and only if} \quad S^2(g_1, g_2) = \top_1$$

The following proposition shows a particular case of fully true attribute implications, which is required before presenting the most important result of this section.

Proposition 8. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\&$ is associative and $\swarrow$ is a forcing-implication. Given a subset of attributes $f_1 \in L_1^Y$, we have that:*

$$\|f_1^{\downarrow\uparrow} \Leftarrow f_1\|_{\mathcal{M}} = \top_1$$

Now, we can present the equivalences which allow us to characterize the fully true multi-adjoint attributes implications. Note that, next result is analogous to the one given in Proposition 5 for the residuated case and the Boolean one, that is, Proposition 3.

Theorem 3. *Let $(L_1, \preceq_1)$ be a complete lattice satisfying the descendent chain condition and $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to $(L_1, \preceq_1)$ such that $\&$ is associative, $\swarrow$ is forcing-implication and $x \& \top_1 = x$, for all $x \in L_1$. Given two subsets of attributes $f_1, f_2 \in L_1^Y$, then we have:*

$$\|f_2 \Leftarrow f_1\|_{\mathcal{M}} = \top_1 \quad \textit{if and only if} \quad f_2 \preceq_1 f_1^{\downarrow\uparrow} \quad \textit{if and only if} \quad f_1^{\downarrow} \preceq_1 f_2^{\downarrow}$$

Finally, an illustrate example is introduced.

Example 2. Given the multi-adjoint frame

$$([0,1]_2, \leq, (\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{DL}, \nwarrow_{DL}))$$

where $(\&_{DG}, \swarrow^{DG}, \nwarrow_{DG}), (\&_{DP}, \swarrow^{DP}, \nwarrow_{DP}), (\&_{DL}, \swarrow^{DL}, \nwarrow_{DL})$ are the adjoint triples defined from the discretization of the Gödel, product and Łukasiewicz t-norms [10], respectively, on $[0,1]_2 = \{0, 0.5, 1\}$ and (X, Y, I, σ) be a context such that the set of objects is $X = \{x_1, x_2, x_3\}$, the set of attributes is $Y = \{y_1, y_2, y_3, y_4, y_5\}$, and the relation $I\colon X \times Y \to [0,1]_2$ and the mapping σ are given in Table 2.

I	y_1	y_2	y_3	y_4	y_5
x_1	0.5	0.5	1	1	1
x_2	1	1	1	0.5	0.5
x_3	0	0.5	0.5	0	1

σ	y_1	y_2	y_3	y_4	y_5
x_1	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$	$\&_{DL}$
x_2	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$	$\&_{DP}$
x_3	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$	$\&_{DG}$

Table 2. Relation I and mapping σ of Example 2.

Consider the fuzzy subset of attributes $f_1 = \{1/y_1, 0.5/y_2, 0.5/y_3, 1/y_4, 0.5/y_5\}$ and we will compute its closure, that is $f_1^{\downarrow\uparrow}$ in order to check if the hypothesis required in Theorem 3 are satisfied. Applying the definition of the operator $^{\downarrow}$, we have that:

$$f_1^{\downarrow}(x_1) = \inf\{I(x_1, y) \nwarrow_{DL} f_1(y) \mid y \in Y\} = 0.5$$
$$f_1^{\downarrow}(x_2) = \inf\{I(x_2, y) \nwarrow_{DP} f_1(y) \mid y \in Y\} = 0.5$$
$$f_1^{\downarrow}(x_3) = \inf\{I(x_3, y) \nwarrow_{DG} f_1(y) \mid y \in Y\} = 0$$

Then, we obtain that $f_1^{\downarrow} = \{0.5/x_1, 0.5/x_2\}$. Applying the operator $^{\uparrow}$ to the set $f_1^{\downarrow}$, we obtain that:

$$f_1^{\downarrow\uparrow}(y_1) = \inf\{I(x_1, y_1) \swarrow^{DL} f_1^{\downarrow}(x_1), I(x_2, y_1) \swarrow^{DP} f_1^{\downarrow}(x_2), I(x_3, y_1) \swarrow^{DG} f_1^{\downarrow}(x_3)\} = 1$$
$$f_1^{\downarrow\uparrow}(y_2) = \inf\{I(x_1, y_2) \swarrow^{DL} f_1^{\downarrow}(x_1), I(x_2, y_2) \swarrow^{DP} f_1^{\downarrow}(x_2), I(x_3, y_2) \swarrow^{DG} f_1^{\downarrow}(x_3)\} = 1$$
$$f_1^{\downarrow\uparrow}(y_3) = \inf\{I(x_1, y_3) \swarrow^{DL} f_1^{\downarrow}(x_1), I(x_2, y_3) \swarrow^{DP} f_1^{\downarrow}(x_2), I(x_3, y_3) \swarrow^{DG} f_1^{\downarrow}(x_3)\} = 1$$
$$f_1^{\downarrow\uparrow}(y_4) = \inf\{I(x_1, y_4) \swarrow^{DL} f_1^{\downarrow}(x_1), I(x_2, y_4) \swarrow^{DP} f_1^{\downarrow}(x_2), I(x_3, y_4) \swarrow^{DG} f_1^{\downarrow}(x_3)\} = 1$$
$$f_1^{\downarrow\uparrow}(y_5) = \inf\{I(x_1, y_5) \swarrow^{DL} f_1^{\downarrow}(x_1), I(x_2, y_5) \swarrow^{DP} f_1^{\downarrow}(x_2), I(x_3, y_5) \swarrow^{DG} f_1^{\downarrow}(x_3)\} = 1$$

Hence, we have that $f_1^{\downarrow\uparrow} = \{1/y_1, 1/y_2, 1/y_3, 1/y_4, 1/y_5\}$. It is clear that $f_2 \leq f_1^{\downarrow\uparrow}$ for all $f_2 \in [0,1]_2^Y$. Applying Theorem 3, we have that any attribute implication whose antecedent is f_1 will be fully true in the given context. In particular, the six non-trivial implications $f_2 \Leftarrow f_1$, obtained from mappings f_2 greater than f_1.

As a conclusion, Theorem 3 facilitates the identification of fully true implications in a given context, since it does not require the computation of the degree of validity based on the whole set of intents of the concepts of the lattice. □

4 Conclusions and future work

This paper has analyzed the fully true implications from two useful fuzzy frameworks, residuated concept lattices and multi-adjoint concept lattices. We have shown that the residuated case contains sufficient properties to ensure the generalization of the Boolean equivalences recalled in Proposition 3. On the other hand, sufficient conditions are needed to ensure a similar equivalence in the flexible multi-adjoint framework. We have proved that, if associative adjoint conjunctors and forcing implications are considered in the adjoint triples, then similar equivalences arise. This result also shows the relevant properties in the residuated framework to the characterization of the fully true residuated attribute implications. In the future, more properties will be studied and a deeper relationship among the definition of validity in the residuated and multi-adjoint frameworks will be examined.

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