# Extended permutations dependent Choquet-like operator and application*

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**Abstract.** In this contribution, we extend the concept of the permutations dependent Choquet-like operator studied in [3] to real-valued inputs. We study its properties and apply this new aggregation in the image inpainting. We propose a modification of the derivative-based approach of the image inpainting introduced by Bertalmio et al. in [5]. We keep the original iterative inpainting procedure, but we replace the image gradient and the Laplacian with the mentioned Choquet-like operator.

**Keywords:** Choquet integral · Conditional aggregation operator · Permutations · Image inpainting · Anisotropic diffusion.

## 1 Introduction

The concept of the Choquet integral as a consequence of the groundbreaking work [9] of Gustav Choquet from 1954, is still studied and developed by many researchers. Over the years this concept was generalized in several ways and applied in many areas such as decision making processes [10], computer vision [8], industry [4], etc. In this contribution, we shall follow the permutations dependent Choquet-like operator, see [3]. The main idea of this generalization is to consider any permutation of the basic set in the formula, not only the permutation of the basic set that reorders the input vector in nondecreasing order (the idea of the standard Choquet integral). The permutations dependent operator is interesting from both an application and a theoretical point of view. It is computationally efficient, and it turns out to be a good technique in edge detection, for more details we refer to [3]. Moreover, it covers several well-known fuzzy integrals such as the Choquet integral, two-fold integral [16], (MC) integral [17], $\mathrm{C}_{\mathrm{Ag}}$ operator [7], etc. Originally, the concept of permutations dependent operator was introduced only for nonnegative vectors. However, in many applications, among them image inpainting, the extension to real-valued vectors is needed. In this contribution, we present one possible way how to extend this operator to a bipolar scale $(-\infty, \infty)$.

In this contribution, we also aim to use our new aggregation technique in image inpainting. It is a conservation process where damaged, or missing parts

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of an artwork are recovered into a complete image. At the beginning of the new millennium, Bertalmio et al. in [5] introduced the first modern effective, and efficient digital image inpainting. This method is based on the derivation of the image, and it is inspired by the standard restoration approach. It uses the concept of isophotes (curves that connect areas of the same brightness) together with anisotropic diffusion. In this contribution, we use our new aggregation technique in the image inpainting. We replace the gradient, the Laplacian, and the isophotes with extended permutations dependent Choquet-like operator. The idea of perceiving fuzzy integrals as a gradient analogy was already published in [12] and further studied in [2], but for edge detection.

The paper is organized as follows. Firstly, in Section 2 we introduce the motivation, basic notations, and definitions. In Section 3 we extend the definition of permutations dependent Choquet-like operator to real-valued vectors. We deal with its properties necessary for image inpainting. In Section 4 we propose our image inpainting method based on the mentioned extension of the Choquet-like operator. Further, we experimentally test the proposed method.

## 2 Basic notations and motivation

Every digital image consists of a finite number of pixels. Therefore, for the needs of the application, we restrict ourselves to a finite space. Let us denote

$$[n] := \{1, 2, \dots, n\}, \quad n \in \mathbb{N},$$

with $\mathbb{N} := \{1, 2, 3, \dots\}$. We shall work with permutations of basic set $[n]$, i.e. bijective mappings $\psi\colon [n] \to [n]$. The set of all permutations of $[n]$ we shall denote by $\mathrm{Perm}([n])$. The powerset of $[n]$ we denote by $2^{[n]}$. By $B^c$ we mean the complement of a set $B \in 2^{[n]}$, i.e. $B^c = [n] \setminus B$. A set function $\mu\colon 2^{[n]} \to [0, \infty)$ such that $\mu(B) \le \mu(C)$ whenever $B \subseteq C$, and $\mu(\emptyset) = 0$ is called a *monotone measure* on $2^{[n]}$. The set of all monotone measures on $2^{[n]}$ with $\mu([n]) > 0$ we denote by $\mathbf{M}$. Under *capacity* we mean $\mu \in \mathbf{M}$ such that $\mu([n]) = 1$. The set of all capacities on $2^{[n]}$ we denote by $\mathbf{M}^1$. By $\mathbf{F}^+$ we denote the set of all nonnegative vectors, i.e. $\mathbf{x}\colon [n] \to [0, \infty)$.

A map $\mathsf{A}(\cdot|B)\colon \mathbf{F}^+ \to [0, \infty)$ is called a *conditional aggregation operator* (CAO for short) with respect to a set $B \in 2^{[n]} \setminus \{\emptyset\}$, if

(i) $\mathsf{A}(\mathbf{x}|B) \le \mathsf{A}(\mathbf{y}|B)$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{F}^+$ such that $x_i \le y_i$ for any $i \in B$,
(ii) $\mathsf{A}(\mathbf{1}_{B^c}|B) = 0$.

Moreover, $\mathsf{A}(\cdot|\emptyset) = 0$ by convention, see [6]. Some examples of CAOs are the *zero* operator $\mathsf{A}^{\mathrm{zero}}(\mathbf{x}|B) = 0$, the *minimum* $\mathsf{A}^{\min}(\mathbf{x}|B) = \min_{i\in B} x_i$, the *maximum* $\mathsf{A}^{\max}(\mathbf{x}|B) = \max_{i\in B} x_i$, also the *sum* $\mathsf{A}^{\mathrm{sum}}(\mathbf{x}|B) = \sum_{i\in B} x_i$, the *arithmetic mean* $\mathsf{A}^{\mathrm{mean}}(\mathbf{x}|B) = \frac{1}{|B|}\sum_{i\in B} x_i$, the *projection* $\mathsf{A}^{\mathrm{proj}}(\mathbf{x}|\{i\}) = x_i$, $i \in [n]$, fuzzy integrals, as the *Choquet* integral, or the *Sugeno* integral, the *Shilkret* integral, see [6]. Under SCA we understand *a sequence of conditional aggregation operators*, i.e. $\mathsf{A} := (\mathsf{A}_i)_1^n$, where $\mathsf{A}_i(\cdot) = \mathsf{A}(\cdot|B_i)$ for each $i \in [n]$. E.g. let $(\mathsf{A}_i)_1^3$ be SCA with $\mathsf{A}_1 = (\mathsf{A}^{\min}(\cdot|\{1, 2\})$, $\mathsf{A}_2 = (\mathsf{A}^{\max}(\cdot|\{1, 2\})$, $\mathsf{A}_3 = (\mathsf{A}^{\min}(\cdot|\{3\})$.

| $\Psi^{\text{con}}$ | | | |
|---|---|---|---|
| $\mathbf{z} \in \mathbf{F}^+$ | $\psi_{\mathbf{z}}^{\text{con}} \in \text{Perm}([3])$ | | |
| | $\psi_{\mathbf{z}}^{\text{con}}(1)$ | $\psi_{\mathbf{z}}^{\text{con}}(2)$ | $\psi_{\mathbf{z}}^{\text{con}}(3)$ |
| $(0,0,1)$ | 3 | 1 | 2 |
| $(0,0,2)$ | 3 | 1 | 2 |
| ... | | ... | |
| $(8,3,7)$ | 3 | 1 | 2 |
| $(8,3,8)$ | 3 | 1 | 2 |
| ... | | ... | |

| $\Psi^{\uparrow}$ | | | |
|---|---|---|---|
| $\mathbf{z} \in \mathbf{F}^+$ | $\psi_{\mathbf{z}}^{\uparrow} \in \text{Perm}([3])$ | | |
| | $\psi_{\mathbf{z}}^{\uparrow}(1)$ | $\psi_{\mathbf{z}}^{\uparrow}(2)$ | $\psi_{\mathbf{z}}^{\uparrow}(3)$ |
| $(0,0,1)$ | 1 | 2 | 3 |
| $(0,0,2)$ | 1 | 2 | 3 |
| ... | | ... | |
| $(8,3,7)$ | 2 | 3 | 1 |
| $(8,3,8)$ | 2 | 1 | 3 |
| ... | | ... | |

| $\Psi$ | | | |
|---|---|---|---|
| $\mathbf{z} \in \mathbf{F}^+$ | $\psi_{\mathbf{z}} \in \text{Perm}([3])$ | | |
| | $\psi_{\mathbf{z}}(1)$ | $\psi_{\mathbf{z}}(2)$ | $\psi_{\mathbf{z}}(3)$ |
| $(0,0,1)$ | 2 | 3 | 1 |
| $(0,0,2)$ | 3 | 1 | 2 |
| ... | | ... | |
| $(8,3,7)$ | 2 | 1 | 3 |
| $(8,3,8)$ | 1 | 2 | 3 |
| ... | | ... | |

**Table 1:** Examples of sets of permutations associated with vectors

Further, we briefly present some motivations for the recently introduced concept in [3]. In the literature, we can see two approaches to using the permutation of the basic set in the construction of operators: Let $\mathbf{x} \in \mathbf{F}$, $\mu \in \mathbf{M}$.

(i) Permutation of $[n]$ is related to the input vector, e.g. the *Choquet integral*

$$\mathrm{C}(\mathbf{x},\mu) = \sum_{i=1}^{n} x_{\psi^{\uparrow}(i)} \cdot \big(\mu(\{\psi^{\uparrow}(i),\dots,\psi^{\uparrow}(n)\}) - \mu(\{\psi^{\uparrow}(i+1),\dots,\psi^{\uparrow}(n)\})\big)\,,$$

where $\psi^{\uparrow} \in \text{Perm}([n])$ such that $x_{\psi^{\uparrow}(1)} \leq x_{\psi^{\uparrow}(2)} \leq \dots \leq x_{\psi^{\uparrow}(n)}$.

(ii) Permutation of $[n]$ need not be solely related to the input vector. It can be chosen arbitrarily (the same for each vector), e.g. the MCC-integral [11] where the permutation relates to the maximal chain of subsets of $[n]$, or the IOWA operator [21], where it is derived from another companion vector.

In [3] we generalized the idea of a preselected permutation of $[n]$. We consider a "database" where each vector is associated with its own preselected permutation (it may or may not be derived from the vector), i.e.

$$\Psi = \{\psi_{\mathbf{z}} \in \text{Perm}([n]) : \mathbf{z} \in \mathbf{F}^+\}.$$

The set of all such sets we shall denote by $\mathscr{P}$. In accordance with the denotation we have used until now, by $\psi^{\uparrow} \in \Psi^{\uparrow}$ we denote a permutation with the property that it reorders the components of vector in nondecreasing order. Analogously we mean $\Psi^{\downarrow}$. If each vector is associated with the same permutation, we shall use the denotation $\Psi^{\text{con}}$. An example of $\Psi^{\text{con}}$ is $\Psi^{\text{id}} = \{\psi_{\mathbf{z}}^{\text{id}} \in \text{Perm}([n]) : \mathbf{z} \in \mathbf{F}^+\}$ where $\psi_{\mathbf{z}}^{\text{id}}$ is the identity. By $\Psi^{\text{mon}}$ we mean the set of permutations that maintains the monotonicity property, i.e. if $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{x}_{\psi_{\mathbf{x}}} \leq \mathbf{y}_{\psi_{\mathbf{y}}}$, $\mathbf{x},\mathbf{y} \in \mathbf{F}^+$. For better understanding see examples in Table 1.

**Definition 1 (cf [3]).** Let $(\mathsf{A}_i)_{i=1}^n$ be a SCA, $\Psi, \Phi \in \mathscr{P}$. Then the $\mathrm{C}_{\mathsf{A},\Psi,\Phi}$ *operator* of $\mathbf{x} \in \mathbf{F}^+$ w.r.t. $\mu \in \mathbf{M}$ is defined as

$$\mathrm{C}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) = \sum_{i=1}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi}) \cdot \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big)\,,$$

where $\mathbf{x}_{\phi} = (x_{\phi(1)},\dots,x_{\phi(n)})$, $\psi$ is the permutation of $[n]$ corresponding to the vector $(\mathsf{A}_1(\mathbf{x}_{\phi}),\dots,\mathsf{A}_n(\mathbf{x}_{\phi}))$ and $E_{\psi(i)} = \{\psi(i),\psi(i+1),\dots,\psi(n)\}$ for each $i \in [n]$ with the convention $E_{\psi(n+1)} = \emptyset$.

In the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator we use two sets $\Psi, \Phi \in \mathscr{P}$, and both play different role. The set $\Psi$ relates to vectors $\mathbf{x}_{\mathsf{A}} = (\mathsf{A}_1(\mathbf{x}_\phi), \dots, \mathsf{A}_n(\mathbf{x}_\phi))$, and the set $\Phi$ is only related to the input vector $\mathbf{x}$. Similar idea appears e.g. in the construction of TOWA operator [22].

**Image inpainting** By *image* we mean a discrete function $I\colon D \to H$, where $D = [r] \times [c]$ represents rows and columns of pixels, $r, c \in \mathbb{N}$, and $H$ represents the colours of pixels. For a colour image $H = \{0, \dots, 255\}^3$ (RGB scale), and for an image in grayscale $H = \{0, \dots, 255\}$ (all three components of the RGB scale have the same value). Thus each pixel of the image is represented by coordinates $[i, j] \in D$, and colour $I(x, y) \in H$. In this contribution, we formally describe methods for grayscale images. For colour images, we repeat this procedure for each RGB component. Further, we briefly describe the method proposed by Bertalmio et al. in [5], which ideas will serve as the basis of our proposed method.

**Bertalmio et al. inpainting method** Let $[x, y] \in D$ and $N_{x,y}$ denotes its neighboring pixels, i.e. $N_{x,y} = \{[u, v] \in D : \exists\, i, j \in \{-1, 0, 1\}, [u + i, v + j] = [x, y]\} \setminus \{[x, y]\}$. Let $\Omega \subset D$ denote the region of the image to be inpainted, and $\partial\Omega$ its boundary, i.e. $\partial\Omega = \{[x, y] \in \Omega : \exists\, [u, v] \in N_{x,y}, [u, v] \notin \Omega\}$. Proposed image inpainting is the iterative method, where only pixels inside $\Omega$ are modified. Every few iterations, a step of anisotropic diffusion is applied for better estimation of isophotes. Let $I^n$ stand for each one of the image pixels of the inpainted area $\Omega$ at the iteration step $n$. Thus $I^0$ is the input image, and by the proposed algorithm, the iteration process is given by the equation

$$I^{n+1}(i, j) = I^n(i, j) + \Delta t \cdot I^n_t(i, j) \tag{1}$$

for any $(i, j) \in \Omega$, where $\Delta t$ is the rate of improvement, and $I^n_t(i, j)$ means update of image $I^n(i, j)$. The step $I^n_t(i, j)$ using smoothness $L^n(i, j)$ is estimated by the Laplacian operator. The change of smoothness is propagated from outside to the $\partial\Omega$ in direction $\overrightarrow{N^n}(i, j)$. Authors proposed formula

$$I^n_t(i, j) = \overrightarrow{\delta L^n}(i, j) \cdot \frac{\overrightarrow{N^n}(i, j)}{|\overrightarrow{N^n}(i, j)|} \cdot |\nabla I^n(i, j)|,$$

where $\overrightarrow{\delta L^n}(i, j)$ expresses the change in smoothness $L^n(i, j)$, and the fraction $\overrightarrow{N^n}(i, j)/|\overrightarrow{N^n}(i, j)|$ expresses the vector orthogonal to the image gradient

$$\frac{\overrightarrow{N^n}(i, j)}{|\overrightarrow{N^n}(i, j)|} = \frac{(-I^n_y(i, j), I^n_x(i, j))}{\sqrt{(I^n_x(i, j))^2 + (I^n_y(i, j))^2 + \varepsilon}}.$$

Finally, $|\nabla I^n(i, j)|$ is the slope-limited version of the norm of the image gradient. Realizing central differences would make the scheme unstable, and this is the reason for using slope-limiters. Within a given iteration, equation (1) is applied to $\partial\Omega$, which is subsequently subtracted from $\Omega$, i.e. $\Omega := \Omega \setminus \partial\Omega$. Then equation (1) is applied to reduced $\Omega$, the boundary $\partial\Omega$ is subtracted from $\Omega$, etc., until $\Omega = \emptyset$. Then the next iteration follows with the same procedure and the original $\Omega$. The algorithm ends when changes in the image are below a given threshold. Authors set $\Delta t = 0.1$, and performed 15 steps of inpainting by equation (1), then 2 steps of anisotropic diffusion (see [18]), again 15 steps of inpainting, and so on.

Let us note that the gradient at a given inpainted pixel is calculated with respect to values of its four neighboring pixels, which may also be from the inpainted area $\Omega$. We propose inpainted method that takes into account all neighboring values of an inpainted pixel, that are not in $\Omega$.

## 3 On the extension of permutations dependent Choquet-like operator

Because of the needs of the application (inpainting), we present the extension of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator to a bipolar scale. We shall extend the above-mentioned construction to real-valued vectors, i.e. $\mathbf{x} = (x_1, \dots, x_n)$, $x_i \in \mathbb{R} = (-\infty, \infty)$, $i \in [n]$. The set of these vectors we shall denote by $\mathbf{F}$. The set of permutations of $[n]$ corresponding to all vectors $\mathbf{z} \in \mathbf{F}$ we shall denote by $\Psi$, i.e.

$$\Psi = \{\phi_{\mathbf{z}} \in \mathrm{Perm}([n]) : \mathbf{z} \in \mathbf{F}\}.$$

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{F}$. By $\mathbf{x}^+$, and $\mathbf{x}^-$ we mean vectors $\mathbf{x}^+ = (x_1^+, \dots, x_n^+)$, and $\mathbf{x}^- = (x_1^-, \dots, x_n^-)$ with $x_i^+ = \max\{x_i, 0\}$, and $x_i^- = \max\{-x_i, 0\}$, $i \in [n]$. Thus $\mathbf{x}^+, \mathbf{x}^- \in \mathbf{F}^+$, and $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$. Further, for the definition of extended permutations dependent Choquet-like operator, is necessary the term a *sequence of ordered pairs of conditional aggregation operators* (oSCA for short) $((\mathsf{A}^1, \mathsf{A}^2)_i)_1^n$,

$$(\mathsf{A}^1, \mathsf{A}^2)_i := (\mathsf{A}^1(\cdot|B_i), \mathsf{A}^2(\bullet|B_i))$$

for each $i \in [n]$. Let us note, that as members of oSCA we allow to consider different or the same CAOs w.r.t. the same conditional set $B \in 2^{[n]}$, e.g. $(\mathsf{A}^1, \mathsf{A}^2)_i = (\widehat{\mathsf{A}}(\cdot|B), \widetilde{\mathsf{A}}(\bullet|B))$, and $(\mathsf{A}^1, \mathsf{A}^2)_j = (\mathsf{A}(\cdot|B), \mathsf{A}(\bullet|B))$, $i, j \in [n]$, $i \neq j$. For example, let $((\mathsf{A}^1, \mathsf{A}^2)_i)_1^3$ be a oSCA such that

$$\begin{aligned}
&(\mathsf{A}^1, \mathsf{A}^2)_1 = (\mathsf{A}^{\mathrm{mean}}(\cdot|\{1,2\}), \mathsf{A}^{\mathrm{Ch}_m}(\bullet|\{1,2\})),\ m \in \mathbf{M}^1\\
&(\mathsf{A}^1, \mathsf{A}^2)_2 = (\mathsf{A}^{\min}(\cdot|\{1,2,3\}), \mathsf{A}^{\max}(\bullet|\{1,2,3\})),\\
&(\mathsf{A}^1, \mathsf{A}^2)_3 = (\mathsf{A}^{\max}(\cdot|\{1,2\}), \mathsf{A}^{\min}(\bullet|\{1,2\})).
\end{aligned}$$

**Definition 2.** Let $((\mathsf{A}^1, \mathsf{A}^2)_i)_1^n$ be a oSCA, and $\Psi, \Phi \in \mathscr{P}$. Then the $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ *operator* of $\mathbf{x} \in \mathbf{F}$ w.r.t. $\mu \in \mathbf{M}$ is defined as

$$\mathrm{eC}_{\mathsf{A},\Psi,\Phi}(\mathbf{x}, \mu) = \sum_{i=1}^{n} \Big(\mathsf{A}^1(\mathbf{x}^+_{\phi_{\mathbf{x}}}) - \mathsf{A}^2(\mathbf{x}^-_{\phi_{\mathbf{x}}})\Big)_{\psi(i)} \cdot \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big), \quad (2)$$

where $\mathbf{x}_\phi = (x_{\phi_{\mathbf{x}}(1)}, \dots, x_{\phi_{\mathbf{x}}(n)})$, $\psi$ is the permutation of $[n]$ corresponding to the vector $((\mathsf{A}^1(\mathbf{x}^+_{\phi_{\mathbf{x}}}) - \mathsf{A}^2(\mathbf{x}^-_{\phi_{\mathbf{x}}}))_1, \dots, (\mathsf{A}^1(\mathbf{x}^+_{\phi_{\mathbf{x}}}) - \mathsf{A}^2(\mathbf{x}^-_{\phi_{\mathbf{x}}}))_n)$, and $E_{\psi(i)} = \{\psi(i), \psi(i+1), \dots, \psi(n)\}$ for each $i \in [n]$ with the convention $E_{\psi(n+1)} = \emptyset$.

If we consider $((\mathsf{A},\mathsf{A}^{\text{zero}})_i)_1^n$, and $(\mathbf{x},\mu)\in\mathbf{F}^+\times\mathbf{M}$, then $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\mathrm{C}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)$. For the implementation of the $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator into the image inpainting method it is necessary to demand certain of its properties. The main ones are the averaging behavior and the idempotency. Both of them partially relate to the monotonicity property. As we show, the $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator is not monotone with respect to vectors in general.

*Example 1.* Let $\mathbf{x}=(2,-5,-3)$, $\mathbf{y}=(3,-4,4)$, and $((\mathsf{A}^1,\mathsf{A}^2)_i)_1^n$ be a oSCA with $(\mathsf{A}^1,\mathsf{A}^2)_1=(\mathsf{A}^{\max},\mathsf{A}^{\min})$, $(\mathsf{A}^1,\mathsf{A}^2)_2=(\mathsf{A}^{\min},\mathsf{A}^{\max})$, $(\mathsf{A}^1,\mathsf{A}^2)_3=(\mathsf{A}^{\text{proj}},\mathsf{A}^{\text{proj}})$, with $B_1=\{2,3\}$, $B_2=\{2,3\}$, $B_3=\{1\}$. Let $\Psi^\uparrow,\Phi\in\mathscr{P}$ such that $\phi_{\mathbf{x}}(1)=3$, $\phi_{\mathbf{x}}(2)=2$, $\phi_{\mathbf{x}}(3)=1$, and $\phi_{\mathbf{y}}=\phi^\uparrow$, and $\mu\in\mathbf{M}^1$. Then

$$\mathrm{eC}_{\mathsf{A},\Psi^\uparrow,\Phi}(\mathbf{x},\mu)=-5\cdot(\mu([3])-\mu(\{1,3\}))-3\cdot(\mu(\{1,3\})-\mu(\{1\}))+2\cdot\mu(\{1\}),$$
$$\mathrm{eC}_{\mathsf{A},\Psi^\uparrow,\Phi}(\mathbf{y},\mu)=-4\cdot(\mu([3])-\mu(\{1,2\}))+3\cdot(\mu(\{1,2\})-\mu(\{1\}))+4\cdot\mu(\{1\}).$$

Thus, for $\mu(\{1,3\})=1$, $\mu(\{1,2\})=0.1$, and $\mu(\{1\})=0$ we get the result $-3=\mathrm{eC}_{\mathsf{A},\Psi^\uparrow,\Phi}(\mathbf{x},\mu)>\mathrm{eC}_{\mathsf{A},\Psi^\uparrow,\Phi}(\mathbf{y},\mu)=-3.3$, but $\mathbf{x}<\mathbf{y}$.

**Lemma 1.** *Let $((\mathsf{A}^1,\mathsf{A}^2)_i)_1^n$ be a oSCA, and $\mathbf{x},\mathbf{y}\in\mathbf{F}$ such that $\mathbf{x}\le\mathbf{y}$. Then*

$$(\mathsf{A}^1(\mathbf{x}^+)-\mathsf{A}^2(\mathbf{x}^-))_i\le(\mathsf{A}^1(\mathbf{y}^+)-\mathsf{A}^2(\mathbf{y}^-))_i$$

*for any $i\in[n]$.*

*Proof.* From the monotonicity of CAOs we have inequalities $\mathsf{A}^1(\mathbf{x}^+)\le\mathsf{A}^1(\mathbf{y}^+)$ and $\mathsf{A}^2(\mathbf{x}^-)\ge\mathsf{A}^2(\mathbf{y}^-)$ for any $(\mathsf{A}^1,\mathsf{A}^2)_i$, $i\in[n]$. Further $\mathsf{A}(\mathbf{x}^+)-\mathsf{A}(\mathbf{x}^-)\le\mathsf{A}(\mathbf{y}^+)-\mathsf{A}(\mathbf{x}^-)\le\mathsf{A}(\mathbf{y}^+)-\mathsf{A}(\mathbf{y}^-)$ for any $(\mathsf{A}^1,\mathsf{A}^2)_i$, $i\in[n]$. □

**Proposition 1.** *Let $\mathbf{x},\mathbf{y}\in\mathbf{F}$. If $\mathbf{x}\le\mathbf{y}$, then*

$$\mathrm{eC}_{\mathsf{A},\Psi^{\text{con}},\Phi^{\text{mon}}}(\mathbf{x},\mu)\le\mathrm{eC}_{\mathsf{A},\Psi^{\text{con}},\Phi^{\text{mon}}}(\mathbf{y},\mu)$$

*for any $((\mathsf{A}^1,\mathsf{A}^2)_i)_1^n$, and any $\mu\in\mathbf{M}$.*

*Proof.* If $\mathbf{x}\le\mathbf{y}$, then $\mathbf{x}_{\phi_{\mathbf{x}}^{\text{mon}}}\le\mathbf{y}_{\phi_{\mathbf{y}}^{\text{mon}}}$. From Lemma 1 we have that for any $((\mathsf{A}^1,\mathsf{A}^2)_i)_1^n$ it holds $(\mathsf{A}^1(\mathbf{x}^+_{\phi_{\mathbf{x}}^{\text{mon}}})-\mathsf{A}^2(\mathbf{x}^-_{\phi_{\mathbf{x}}^{\text{mon}}}))_i\le(\mathsf{A}^1(\mathbf{y}^+_{\phi_{\mathbf{y}}^{\text{mon}}})-\mathsf{A}^2(\mathbf{y}^-_{\phi_{\mathbf{y}}^{\text{mon}}}))_i$ for any $i\in[n]$. Further, since $\psi^{\text{con}}\in\Psi^{\text{con}}$, then $(\mathsf{A}^1(\mathbf{x}^+_{\phi_{\mathbf{x}}^{\text{mon}}})-\mathsf{A}^2(\mathbf{x}^-_{\phi_{\mathbf{x}}^{\text{mon}}}))_{\psi^{\text{con}}(i)}\le(\mathsf{A}^1(\mathbf{y}^+_{\phi_{\mathbf{y}}^{\text{mon}}})-\mathsf{A}^2(\mathbf{y}^-_{\phi_{\mathbf{y}}^{\text{mon}}}))_{\psi^{\text{con}}(i)}$ for any $i\in[n]$. □

*Remark 1.* The assumptions of the previous proposition are satisfied for the sets of permutations $\Phi^\uparrow,\Phi^\downarrow,\Phi^{\text{con}}\in\mathscr{P}$.

In the following, we describe assumptions under which $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator is an averaging type of operator. This property ensures the natural requirement that the inpainted colour is within the range of neighboring colours.

**Proposition 2.** *Let $((\mathsf{A}^1,\mathsf{A}^2)_i)_1^n$ be a oSCA. If for any $(\mathsf{A}^1,\mathsf{A}^2)_i$, $i\in[n]$, it holds $\min_{i\in[n]}z_i\le\mathsf{A}^k(\mathbf{z})\le\max_{i\in[n]}z_i$ for any $\mathbf{z}\in\mathbf{F}^+$, $k\in[2]$, then*

$$\min_{i\in[n]}x_i\le\mathrm{eC}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)\le\max_{i\in[n]}x_i$$

*for any $\mathbf{x}\in\mathbf{F}$, $\mu\in\mathbf{M}^1$, and $\Psi,\Phi\in\mathscr{P}$.*

*Proof.* Because of the assumptions we have $\min_{i\in[n]} z_i^+ \le \mathsf{A}^1(\mathbf{z}^+) \le \max_{i\in[n]} z_i^+$, and $-\max_{i\in[n]} z_i^- \le -\mathsf{A}^2(\mathbf{z}^-) \le -\min_{i\in[n]} z_i^-$. Thus we have

$$\min_{i\in[n]} z_i = \min_{i\in[n]} z_i^+ - \max_{i\in[n]} z_i^- \le \mathsf{A}^1(\mathbf{z}^+) - \mathsf{A}^2(\mathbf{z}^-) \le \max_{i\in[n]} z_i^+ - \min_{i\in[n]} z_i^- = \max_{i\in[n]} z_i.$$

Further, $\sum_{i=1}^{n} \Big(\mathsf{A}^1(\mathbf{x}^+_{\phi_\mathbf{x}}) - \mathsf{A}^2(\mathbf{x}^-_{\phi_\mathbf{x}})\Big)_{\psi(i)} \cdot \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big) \le \sum_{i=1}^{n} \max_{j\in[n]} x_j \cdot \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big) = \max_{j\in[n]} x_j \cdot \sum_{i=1}^{n} \mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)}) = \max_{j\in[n]} x_j \cdot \mu([n]) = \max_{j\in[n]} x_j$ for any $\Psi, \Phi \in \mathscr{P}$. Analogously for boundary from below. $\square$

*Remark 2.* Let $\mathbf{z} \in \mathbf{F}$, and $B \subseteq [n]$. Some examples of $(\mathsf{A}^1, \mathsf{A}^2)$ satisfying the assumption of Proposition 2 are

- (i) $(\mathsf{A}^1, \mathsf{A}^2) = (\mathsf{A}^{\min}, \mathsf{A}^{\max})$. In fact, $\mathsf{A}^{\min}(\mathbf{z}^+|B) - \mathsf{A}^{\max}(\mathbf{z}^-|B)$ expresses the minimum of components of $\mathbf{z}$ with respect to $B$,
- (ii) $(\mathsf{A}^1, \mathsf{A}^2) = (\mathsf{A}^{\max}, \mathsf{A}^{\min})$. In fact, $\mathsf{A}^{\max}(\mathbf{z}^+|B) - \mathsf{A}^{\min}(\mathbf{z}^-|B)$ expresses the maximum of components of $\mathbf{z}$ with respect to $B$,
- (iii) $(\mathsf{A}^1, \mathsf{A}^2) = (\mathsf{A}^{\mathrm{mean}}, \mathsf{A}^{\mathrm{mean}})$. In fact $\mathsf{A}^{\mathrm{mean}}(\mathbf{z}^+|B) - \mathsf{A}^{\mathrm{mean}}(\mathbf{z}^-|B)$ expresses the mean of components of $\mathbf{z}$ with respect to $B$.
- (iv) $(\mathsf{A}^1, \mathsf{A}^2) = (\mathsf{A}^{\mathrm{Ch}_m}, \mathsf{A}^{\mathrm{Ch}_m})$, $m \in \mathbf{M}^1$. In fact, $\mathsf{A}^{\mathrm{Ch}_m}(\mathbf{z}^+|B) - \mathsf{A}^{\mathrm{Ch}_m}(\mathbf{z}^-|B)$ expresses the symmetric Choquet integral (or Šípoš integral [20]) of $\mathbf{z}$ with respect to $B$.

The previous proposition can be extended also for any $\mu \in \mathbf{M}$. It is enough to suppose $\frac{1}{\mu([n])} \min_{i\in[n]} z_i \le \mathsf{A}^k(\mathbf{z}) \le \frac{1}{\mu([n])} \max_{i\in[n]} z_i$ for any $\mathbf{z} \in \mathbf{F}^+$.

Let us note, that from Proposition 2 it follows, that if $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator has the averaging behavior property, then it is also idempotent, see Corollary 1. The reverse implication is not true see Corollary 2. Idempotency ensures that if the neighboring pixels are equal, then the inpainted pixel is the same.

**Corollary 1.** *Let* $((\mathsf{A}^1, \mathsf{A}^2)_i)_1^n$ *be a oSCA,* $\Psi, \Phi \in \mathscr{P}$*, and* $\mu \in \mathbf{M}^1$*. If*

$$\min_{i\in[n]} x_i \le \mathrm{eC}_{\mathsf{A},\Psi,\Phi}(\mathbf{x}, \mu) \le \max_{i\in[n]} x_i$$

*for any* $\mathbf{x} \in \mathbf{F}$*, then* $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}((c, \dots, c), \mu) = c$ *for any* $c \in \mathbb{R}$.

*Remark 3.* Examples of oSCA for which the $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator is idempotent can be found in Remark 2.

The following corollary describes conditions under which averaging behavior property is equivalent to idempotency for the $\mathrm{eC}_{\mathsf{A},\Psi,\Phi}$ operator.

**Corollary 2.** *Let* $((\mathsf{A}^1, \mathsf{A}^2)_i)_1^n$ *be a oSCA, and* $\mu \in \mathbf{M}^1$*. The inequality*

$$\min_{i\in[n]} x_i \le \mathrm{eC}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{mon}}}(\mathbf{x}, \mu) \le \max_{i\in[n]} x_i$$

*holds for any* $\mathbf{x} \in \mathbf{F}$ *if and only if* $\mathrm{eC}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{mon}}}((c, \dots, c), \mu) = c$ *for any* $c \in \mathbb{R}$.

**Input:** $I^0$, $\Omega^{\text{orig}}$, $R$, $\Delta t$, $\mu$, $T$
**Output:** $I^R$
$I^0 := \text{anisDiff}(I^0, \lambda, \kappa)$;
**for** each $n \in \{1, \ldots, R\}$ **do**
  $\Omega := \Omega^{\text{orig}}$;
  **while** $\Omega \neq \emptyset$ **do**
    **if** $|N_{x,y} \setminus \Omega| \geq 4$ **then**
      $z_k := I^{n-1}(x,y) - I^{n-1}(x_k, y_k)$, $[x_k, y_k] \in N_{x,y} \setminus \Omega$;
      $n_{x,y} := |N_{x,y} \setminus \Omega|$, $\mathbf{z} := (z_1, \ldots, z_{n_{x,y}})$, $\mathbf{w} := (1, \ldots, 1)$;
      **if** $[2x - x_k, 2y - y_k] \notin \Omega$, $|I(x,y) - I(2x - x_k, 2y - y_k)| \leq T$ **then**
        $w_k := w_k + 1$;
      **end**
      **if** $[2x_k - x, 2y_k - y] \notin \Omega$, $|I(x,y) - I(2x_k - x, 2y_k - y)| \leq T$ **then**
        $w_k := w_k + 1$;
      **end**
      $z^{\min} := \min\{z_1, \ldots, z_{n_{x,y}}\}$, $z^{\max} := \max\{z_1, \ldots, z_{n_{x,y}}\}$;
      **if** $z_i - z^{\min} \leq z^{\max} - z_i$ **then**
        $S^{\max} := S^{\max} \cup \{(z_i, w_i)\}$, $(z_i^{\max}, w_i^{\max}) := (z_i, w_i)$;
      **else**
        $S^{\min} := S^{\max} \cup \{(z_i, w_i)\}$, $(z_i^{\max}, w_i^{\min}) := (z_i, w_i)$;
      **end**
      $\mathbf{z}^{\min} := (z_1^{\min}, \ldots, z_{|S^{\min}|}^{\min})$, $\mathbf{w}^{\min} = (w_1^{\min}, \ldots, w_{|S^{\min}|}^{\min})$;
      $\mathbf{z}^{\max} := (z_1^{\max}, \ldots, z_{|S^{\max}|}^{\max})$, $\mathbf{w}^{\max} = (w_1^{\max}, \ldots, w_{|S^{\max}|}^{\max})$;
      $c^{\min} := 2/(\min\{w_1^{\min}, \ldots, w_{|S^{\min}|}^{\min}\} + \max\{w_1^{\min}, \ldots, w_{|S^{\min}|}^{\min}\})$;
      $c^{\max} = 2/(\min\{w_1^{\max}, \ldots, w_{|S^{\max}|}^{\max}\} + \max\{w_1^{\max}, \ldots, w_{|S^{\max}|}^{\max}\})$;
      $\mathbf{s}^{\min} := \mathbf{z}^{\min} \cdot (c^{\min} \cdot \mathbf{w}^{\min})$, $\mathbf{s}^{\max} := \mathbf{z}^{\max} \cdot (c^{\max} \cdot \mathbf{w}^{\max})$;
      $I_t^n(x,y) := |N_{x,y} \setminus \Omega| \cdot (\text{eC}_{\mathsf{A},\Psi^{\downarrow},\Phi^{\uparrow}}(\mathbf{s}^{\min}, \mu) + \text{eC}_{\mathsf{A},\Psi^{\downarrow},\Phi^{\uparrow}}(\mathbf{s}^{\max}, \mu))/2$;
      $I^{n+1}(x,y) := I^n(x,y) - \Delta t \cdot I_t^n(x,y)$, $\Omega := \Omega \setminus \partial\Omega$;
    **end**
  **end**
**end**

**Algorithm 1:** Proposed inpainting method

## 4 Image inpainting based on $\text{eC}_{\mathsf{A},\Psi,\Phi}$ operator

In this section, we propose a new method, or algorithm, for image inpainting based on $\text{eC}_{\mathsf{A},\Psi,\Phi}$ operator. As we stated, we keep the notations and ideas of the method proposed by Bertalmio et al. in [5], see Subsection 2. From a computational point of view, the proposed $\text{eC}_{\mathsf{A},\Psi,\Phi}$ operator is time-efficient, what is important when we take into account how many inpainted pixels and iterations we work with. The proposed method, see Algorithm 1, can be described in several steps:

(i) The inputs for proposed algorithm are: the input image $I_0$, the area $\Omega^{\text{orig}}$ for inpainting, the number of iterations $R$, the rate of improvement $\Delta t$, $\mu \in \mathbf{M}^1$, and threshold $T \in \{0, \ldots, 255\}$. The output is the image $I^R$.

(ii) We apply the discretization of the anisotropic diffusion equation introduced by Perona and Malik, see the equation (7) in [18], with $\lambda = 0.25$ and $\kappa$ equals to the approximation of image gradient magnitude.

(iii) For each pixel of $\partial\Omega$ that has at least four neighbors outside we create a vector $\mathbf{z}$ expressing the difference between the inpainted pixel and its neighbours outside $\Omega$. This is an analogy of an image gradient. We will express the importance of individual pixels with appropriate weights $\mathbf{w}$.

(iv) Further we model the isophote concept with weights. If the color difference of centrally symmetrical pixels (both from $\Omega$), or two consecutive pixels (also both from $\Omega$) in the direction of the center pixel is within the range determined by $T$, we increase the weights of these pixels by 1.

(v) In the next step, it is appropriate to divide the components of $\mathbf{z}$ with corresponding weights $\mathbf{w}$ into two groups – those that are closer to the minimum, or to the maximum component of $\mathbf{z}$. Thus we create vectors $\mathbf{z}^{\min}$ with $\mathbf{w}^{\min}$, and $\mathbf{z}^{\max}$ with $\mathbf{w}^{\max}$. We rescale the weights using the procedure below and multiply them with the corresponding vectors $\mathbf{z}^{\min}$, or $\mathbf{z}^{\max}$, respectively.

(vi) The vectors $\mathbf{s}^{\min} = \mathbf{z}^{\min} \cdot \mathbf{w}^{\min}$ and $\mathbf{s}^{\max} = \mathbf{z}^{\max} \cdot \mathbf{w}^{\max}$ are inputs for the aggregation by $\mathrm{eC}_{\mathsf{A},\Psi\downarrow,\Phi\uparrow}$ operator with respect to a capacity $\mu$.

(vii) The final inpainted value is the arithmetic mean of $\mathrm{eC}_{\mathsf{A},\Psi\downarrow,\Phi\uparrow}(\mathbf{s}^{\min},\mu)$ and $\mathrm{eC}_{\mathsf{A},\Psi\downarrow,\Phi\uparrow}(\mathbf{s}^{\max},\mu)$ multiplied by the aliquot numbers of neighbours of an inpainted pixel outside of $\Omega$.

(viii) This value is further multiplied by the constant $\Delta t$ and subtracted from the value of the inpainted pixel from the previous iteration.

(ix) The number of iterations defines the number of repetitions of steps (ii)–(viii). Additionally, every 100 iterations we apply 2 iterations of anisotropic diffusion with the same parameters as above. In some cases, to suppress blurring, after a selected number of iterations, it is appropriate to recolor randomly selected pixels with the maximum, minimum, average pixel, etc. from their surroundings.

In the following, we experimentally apply our proposed image inpainted method described by Algorithm 1. For the purpose of the experiment, we use images from Berkeley Segmentation Dataset and Benchmark (BSDS500), see [1]. We select some images from the dataset, damage and restore them to their original state, or remove unwanted objects using our proposed method. In inpainting algorithm we use $\mathrm{eC}_{\mathsf{A},\Psi\downarrow,\Phi\uparrow}$ operator with oSCA $(\mathsf{A}^1,\mathsf{A}^2)_1^n$ described in Table 2. From the definition of the $\mathrm{eC}_{\mathsf{A},\Psi\downarrow,\Phi\uparrow}$ operator, the number of members of oSCA is equal to the dimension of the aggregated vector. The dimension can take values from 1 to 8, see step (v) of Algorithm 1. For this reason, it is always necessary to consider only the aliquot part of the ordered pairs listed in Table 2. Further, as the monotone measure we use the *power measure* $\mu \in \mathbf{M}$ defined as

$$\mu(B) = \left(\frac{|B|}{n}\right)^q$$

| $i$ | $(\mathsf{A}^1, \mathsf{A}^2)_i$ | $B_i$ |
|---|---|---|
| 1 | $(\mathsf{A}^{\text{mean}}, \mathsf{A}^{\text{mean}})$ | $[n]$ |
| 2 | $(\mathsf{A}^{\text{Ch}m}, \mathsf{A}^{\text{Ch}m})$ | $[n]$ |
| 3 | $(\mathsf{A}^{\text{mean}}, \mathsf{A}^{\text{mean}})$ | $\{\frac{n+1}{2}\}$, if $n$ is odd, and $\{\frac{n}{2}, \frac{n}{2}+1\}$, if $n$ is even |
| 4 | $(\mathsf{A}^{\text{mean}}, \mathsf{A}^{\text{mean}})$ | $\{\lfloor 0.25 \cdot n \rfloor, \lfloor 0.75 \cdot n \rfloor\}$ |
| 5 | $(\mathsf{A}^{\text{maxMin}}, \mathsf{A}^{\text{maxMin}})$ | $\{1, n\}$ |
| 6 | $(\mathsf{A}^{\text{Ch}m}, \mathsf{A}^{\text{Ch}m})$ | $\{1, 2, n-1, n\}$ |
| 7 | $(\mathsf{A}^{\text{maxMin}}, \mathsf{A}^{\text{maxMin}})$ | $\{2, n-1\}$ |
| 8 | $(\mathsf{A}^{\text{maxMin}}, \mathsf{A}^{\text{maxMin}})$ | $\{3, n-2\}$ |

**Table 2:** The oSCA used in the image inpainting with respect to the dimension $n$ of the aggregated vector

for any $B \subseteq [n]$, with $q = 0.9$. Figure 1 shows a sample of the above-mentioned images from the BSDS500 dataset and the subsequent use of our proposed image inpainting method to remove unwanted elements of these images.

The inpainting of images can be evaluated both qualitatively and quantitatively. As we see in Figure 1, a better result is achieved with respect to a small inpainted area $\Omega$, or with respect to the neighborhood of the area $\Omega$ with the same shade. For example, let us consider the damaged image 385028 from BSDS500 in Figure 1 and the letter “a” in the word “adipiscing”. In Figure 2 one can see the diagram that shows the original pixel values (in a shade of gray before the image damage) from the area bounded by the mentioned letter “a” in blue, and the values of the inpainted pixels in orange. As we can see the inpainted values copy the original pixel values well except for the areas where the inpainted area hid the additional information for the image inpainting method. From the above, it is possible to deduce the fact that the concept of additive measures and integrals is also applicable in this area of image processing and in a certain way replaces the gradient and Laplacian.

## 5 Conclusion

In this contribution, we have presented an extension of the permutations dependent operator to a bipolar scale. We have used this new aggregation in the inpainting problem where this extension was necessary. In the contribution we have used the asymmetric extension, in the future it would be interesting to compare the result with other extensions using the idea of construction of the symmetric Choquet integral called also the Šipoš integral [20], the balancing Choquet integral [15], fusion Choquet integral [13], or BIOWA operators [14].

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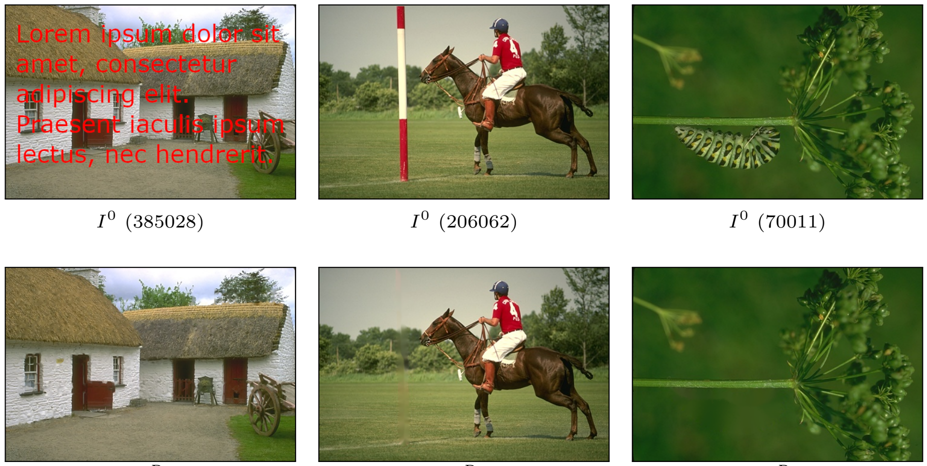


$I^0$ (385028) $I^0$ (206062) $I^0$ (70011)

$I^R$ ($R = 480$, $\Delta t = 0.01$, $T = 40$)

$I^R$ ($R = 8000$, $\Delta t = 0.01$, $T = 40$)

$I^R$ ($R = 3000$, $\Delta t = 0.01$, $T = 40$)

**Fig. 1:** Image inpainting of 385028, 206062, and 35028 images from BSDS500


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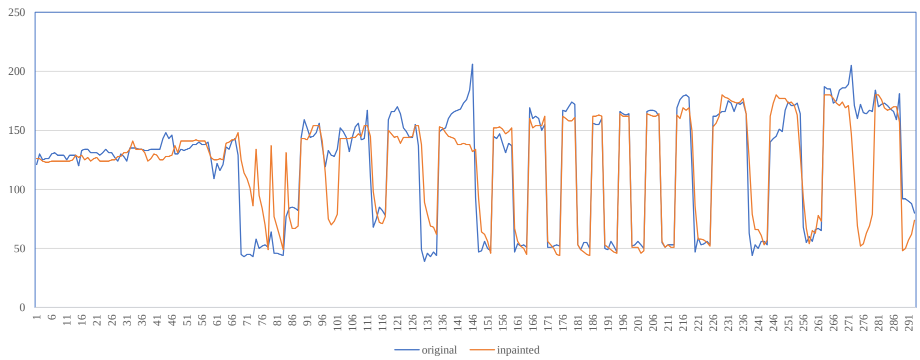


**Fig. 2:** Comparison of inpainted pixels with original ones


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