Existence of Quasi-Nash Equilibrium in Game with Discontinuous Payoff Functions

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Abstract. In the paper a convex game with discontinuous payoff functions is considered. This paper introduces the concept of quasi-Nash equilibrium, which allows us to analyze games with discontinuous payoff functions by approximating them with continuous and concave functions. We show that if the approximation is close enough, then the quasi-Nash equilibrium is close to the true Nash equilibrium (if it exists).

Keywords: Nash equilibrium · Convex game · Discontinuity of payoff function · Quasi-equilibrium.

1 Introduction

Nash equilibrium is regarded as one of the most important notions in Game Theory. The concept dates back to at least Cournout [9]. However, its current formalization is due to Nash, whose original proof [18], given in 1950, relies on Kakutani's fixed point theorem. One year later, Nash [19] gave a different proof, which uses Bohl-Brouwer fixed point theorem ([3], [4]). Nash equilibrium is a strategy profile where each agent is reacting optimally to other players' plans. In discontinuous games, Kakutani's theorem cannot be directly applied because a player may not have an optimal response or their best response may not be a continuous mapping depending on the choices of other players.

One can find quite a lot of literature dealing with discontinuous payoffs functions. One such article is [1]. In this article it has been proven an equilibrium existence theorem for games with discontinuous payoffs and convex and compact strategy spaces. It generalizes the classical results of [26] and [17]. Authors of [1] show that a condition on the payoffs, named continuous security, is sufficient for existence of equilibrium in games with convex and compact strategy spaces. The authors of [8] generalize [26] and [1] results using the notion of surrogate better-reply security for discontinuous skew-symmetric games. The author of the article [26] himself has repeatedly improved and supplemented his results, see [28] for a summary. In the above articles and many more ([2, 10, 11, 14, 20, 21, 24, 25] and others) games with discontinuous payoffs functions the conditions under which the Nash equilibrium exists are found.

Also recent textbooks (e.g. [15]) discuss the existence of Nash equilibria in discontinuous games and approximate equilibria in discontinuous games.

Not so much literature can be found in which discontinuous payoff functions are approximated by certain types of other functions with good properties (however, such articles exist, e.g. [27]).

The paper is organized as follows. In Section 2, we first give some concepts, definitions and theorems used throughout the article. In Section 3, we will present the concept of quasi-Nash equilibrium and the possibility of obtaining it in games with discontinuous payoff functions. Finally, we give some examples and ideas for future work.

2 Existence of Nash equilibrium

Let $N = \{1, 2, ..., n\}$ be the finite set of players. Each player $i \in N$ has a pure strategy set S_i. In general case, the sets S_i may possess any structure (a finite set of elements, a subset of $\mathbf{R}^n$, etc.). As a result, player i obtain the payoffs u_i.

Definition 1 ([16]). *A normal-form game is an object*

$$\Gamma = \{N,\, S_1, ..., S_n,\, u_1, ..., u_n\},$$

where S_i designates the sets of strategies of players $i \in N$ and u_i indicates their payoff functions, $u_i : S = S_1 \times S_2 \times ... \times S_n \to \mathbf{R}$, $i \in N$.

We define the strategies that maximize a player's payoff, while fixing the combination of all other players' strategies. $s_i \in S_i$ denotes one strategy of player i, while $s_{-i} \in S_{-i} = S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_n$ represents one strategy combination from all other players without player i.

Definition 2 ([12]). *For any player $i \in N$ and any strategy combination by i's opponents, $s_{-i} \in S_{-i}$, a best response of player i to s_{-i} is a strategy $s_i^* \in S_i$ such that*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_i \in S_i$.

In general, even though the strategy combination by the opponents is fixed, there can be multiple best responses for player i. Thus we define the set of best responses to s_{-i}:

$$BR_i(s_{-i}) = \{s_i^* \in S_i \,|\, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}),\, \forall s_i \in S_i\}.$$

Since all players are rational, all players should be playing a best response to the rest of the strategy combination.

Definition 3 ([12]). *A strategy combination $(s_1^*, s_2^*, ..., s_n^*) \in S$ is a Nash equilibrium if, for any player $i \in N$*

$$s_i^* \in BR_i(s_1^*, s_2^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*).$$

Alternatively, $(s_1^*, s_2^*, ..., s_n^*) \in S$ satisfies the inequalities

$$\forall i \in N \,\forall s_i \in S_i \quad u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Nash equilibrium does not exist in all normal-form games. We will consider a class of convex games where equilibria exist.

Definition 4 ([16]). *A function* $u : X \to \mathbf{R}$*,* $X \subset \mathbf{R}^n$*, is called concave on a set* X*, if for any* $x, y \in X$ *and* $\alpha \in [0, 1]$ *the inequality holds true*

$$u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y).$$

This definition directly implies the inequality for concave functions

$$u(\sum_{i=1}^{k} \alpha_i x_i) \geq \sum_{i=1}^{k} \alpha_i u(x_i)$$

for any convex combination of the points $x_i \in X$, $i = 1, ..., k$, where $\alpha_i \geq 0$, $i = 1, ..., k$, and $\sum_{i=1}^{k} \alpha_i = 1$.

Definition 5 ([16]). *A normal-form game* $\Gamma = \{N, S_1, ..., S_n, u_1, ..., u_n\}$ *is called a convex game if for all* $i \in N$ *satisfy the conditions*
1) S_i *is a nonempty, convex, compact subset of the space* $\mathbf{R}^m$*,*
2) u_i *is a continuous function on the set* $S = S_1 \times S_2 \times ... \times S_n$*,*
3) for all fixed $s_{-i} \in S_{-i}$*, the payoff function* $u_i(s_i, s_{-i})$ *is concave by the corresponding variable* s_i*.*

If a function is concave in a convex set, then it is continuous in every inner point of the set ([22]). A weaker condition than concavity is quasi-concavity.

Definition 6 ([2, 15]). *A function* $u : X \to \mathbf{R}$*,* $X \subset \mathbf{R}^n$*, is called quasi-concave on a set* X*, if the level sets* $\{x \mid u(x) \geq \alpha\}$ *are convex for all reals* α*.*

Any concave function is quasi-concave, but not vice versa. A quasi-concave function can also be a discontinuous function.

The Nash theorem ([19]) gives a central statement regarding equilibrium existence in convex games.

Theorem 1 ([16, 19]). *Any convex game has a Nash equilibrium.*

The game is quasi-concave if for every player $i \in N$, S_i is convex and for every $s_{-i} \in S_{-i}$ the mapping $u_i(\cdot, s_{-i})$ is quasi-concave, The game is continuous if for every $i \in N$, u_i is a continuous function. In this terminology, the following result holds.

Theorem 2 ([13, 2]). *Every continuous, quasi-concave, and compact game admits a Nash equilibrium.*

Note that the existence of Nash equilibria in the duopolies relates to the form of payoff functions ([16], all economic examples considered employ continuous concave functions).

Example 1. We consider the Cournot duopoly (formulated by Cournot [9], well before Nash work, see [12, 16, 23] and other game theory or microeconomics textbooks). The term “duopoly” corresponds to a two-player game.

Assume that in the market for a certain good, there are only two producers, called firm 1 and firm 2. Each firm chooses a quantity q_1 and q_2 of the good to supply to the market. In this model, the quantities represent the strategies of the players. The set of strategies (common to both firms) is the set of all non-negative real numbers $S_1 = S_2 = [0, +\infty[$. The payoff of a firm is its profit, which is the revenue minus the cost of production. The market price of the product equals an initial price A after deduction of the total quantity $Q = q_1 + q_2$. And so, the unit price constitutes $A - Q$. The firms' unit production costs are c_1 and c_2, respectively. The coefficient c_i, $i = 1, 2$, is the marginal cost such that, for an additional one unit of production, firm i incurs an additional cost of c_i. We also assume that $A > \max\{c_1, c_2\}$. Consequently, the players' payoffs take the form

$$\begin{aligned} u_1(q_1, q_2) &= (A - q_1 - q_2)q_1 - c_1 q_1 = -q_1^2 + (A - q_2 - c_1)q_1, \\ u_2(q_1, q_2) &= (A - q_1 - q_2)q_2 - c_2 q_2 = -q_2^2 + (A - q_1 - c_2)q_2. \end{aligned} \tag{1}$$

The payoff functions (1) are quadratic functions of q_1 and q_2, respectively. They are continuous and concave. If we assume that only a limited quantity of the good can be produced, no larger than K, then $S_1 = S_2 = [0, K]$, and we get a convex game in which a Nash equilibrium exists.

The payoff functions attain the maximum at q_1 and q_2 where the derivatives of u_1 and u_2 is 0, respectively,

$$\begin{aligned} (u_1)'_{q_1} &= -2q_1 + A - q_2 - c_1, \\ (u_2)'_{q_2} &= -2q_2 + A - q_1 - c_2. \end{aligned} \tag{2}$$

When firm 2 strategy is q_2, firm 1 the best response is

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

Similarly, When firm 1 strategy is q_1, firm 2 the best response is

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

By resolving the derived system of equations (2), we find the Nash equilibrium

$$q_1^* = \frac{1}{3}(A + c_2 - 2c_1), \quad q_2^* = \frac{1}{3}(A + c_1 - 2c_2).$$

■

It is convenient to assume that the payoff function is continuous because mathematical tools are well developed for the analysis of continuous processes. But in the general case, price is a discrete quantity, and most goods can only be sold in whole numbers. It is convenient to treat prices and quantities of goods as continuous quantities when a very large number of goods (such as bread) can be produced.

In the case that this good is, e.g. an aeroplane or a power station, its demand is naturally an integer. Obviously, if the good is a piece-good (table, shoes, house and other) then the demand for this good is an integer. Similarly, the supply of piece-goods is an integer. Therefore the demand and supply functions for piece-goods are discontinuous and consequently the payoff function too (see [6, 7, 23]).

3 Existence of quasi-Nash equilibrium

We propose to approximate the payoff function.

Let $u_i : S \to \mathbf{R}$ be a payoff function of player i, $i = 1, ..., n$. It is possible that it may not be continuous and not concave (by the corresponding variable s_i of player i).

Assumption.
$\exists \mu > 0 \quad \exists \overline{u}_i : S \to \mathbf{R}$ – continuous and concave functions by the corresponding variable s_i of player i such that

$$|\overline{u}_i(s) - u_i(s)| \leq \mu, \quad \forall s \in S, \quad \forall i \in N.$$

Definition 7. *A strategy combination* $(\overline{s}_1^*, \overline{s}_2^*, ..., \overline{s}_n^*) \in S$ *is a quasi-Nash equilibrium of normal-form game* $\Gamma = \{N, S_1, ..., S_n, u_1, ..., u_n\}$ *if* $(\overline{s}_1^*, \overline{s}_2^*, ..., \overline{s}_n^*) \in S$ *is a Nash equilibrium of normal-form game* $\overline{\Gamma} = \{N, S_1, ..., S_n, \overline{u}_1, ..., \overline{u}_n\}$.

If the Assumption is satisfied and S_i, $i \in N$, are a nonempty, convex, compact subsets, then payoff functions of players are bounded (see [6] Proposition 3.1) but this does not mean that there exists a point at which the function can reach a maximum or minimum value. Assume that the players aim to achieve the highest possible value of the payoff function.

Theorem 3. *Let* Γ *be normal-form game and* S_i *be a nonempty, convex, compact subset of the space* $\mathbf{R}^m$, $i \in N$. *Let Assumption be fulfilled. Then for every* $i \in N$

$$\exists \max_{s \in S} \overline{u}_i(s) \in \mathbf{R}, \ \exists \sup_{s \in S} u_i(s) \in \mathbf{R} \ \textit{and} \ |\max_{s \in S} \overline{u}_i(s) - \sup_{s \in S} u_i(s)| \leq \mu.$$

Proof. Since S_i is a compact subset of the space $\mathbf{R}^m$, $i \in N$, then S is a compact set. By Assumption $\overline{u}_i : S \to \mathbf{R}$, $i \in N$, are continuous functions, therefore by Weierstrass extreme value theorem $\exists \max_{s \in S} \overline{u}_i \in \mathbf{R}$. Since Assumption is satisfied and S_i, $i \in N$, are compact sets, then payoff functions of players are bounded ([6]), therefore $\exists \sup_{s \in S} u_i(s) \in \mathbf{R}$, $i \in N$.

Let $i \in N$. Suppose

$$|\max_{s \in S} \overline{u}_i(s) - \sup_{s \in S} u_i(s)| > \mu.$$

Two cases are possible. In the first case

$$\max_{s\in S}\overline{u}_i(s)-\sup_{s\in S}u_i(s)>\mu\geq 0.$$

Then $\exists s_0\in S$ such that

$$\max_{s\in S}\overline{u}_i(s)=\overline{u}_i(s_0)>\mu+\sup_{s\in S}u_i(x)\geq\mu+u_i(s_0),\ \forall x\in S.$$

It follows that

$$\overline{u}_i(s_0)-u_i(s_0)>\mu$$

which is contrary to the Assumption.

In the second case

$$\max_{s\in S}\overline{u}_i(s)-\sup_{s\in S}u_i(x)<-\mu\leq 0 \text{ or } \sup_{s\in S}u_i(x)-\max_{s\in S}\overline{u}_i(s)-\mu=\gamma>0.$$

From the definition of supremum it follows that

$$\forall\varepsilon>0\ \exists s'\in S\quad \sup_{s\in S}u_i(s)<u_i(s')+\varepsilon.$$

If $\varepsilon=\frac{\gamma}{2}$, then $\exists s_1\in S$ such that $\sup_{s\in S}u_i(s)<u_i(s_1)+\frac{\gamma}{2}$.

Since $\forall s\in S\ \max_{s\in S}\overline{u}_i(s)\geq\overline{u}_i(s)$ then

$$\begin{aligned}
&\gamma=\sup\nolimits_{s\in S}u_i(x)-\max\nolimits_{s\in S}\overline{u}_i(s)-\mu<\\
&<u_i(s_1)+\tfrac{\gamma}{2}-\max\nolimits_{s\in S}\overline{u}_i(s)-\mu=\\
&=u_i(s_1)-\overline{u}_i(s_1)+\overline{u}_i(s_1)+\tfrac{\gamma}{2}-\max\nolimits_{s\in S}\overline{u}_i(s)-\mu\leq\\
&\leq\mu+\overline{u}_i(s_1)+\tfrac{\gamma}{2}-\max\nolimits_{s\in S}\overline{u}_i(s)-\mu\leq\\
&\leq\max\nolimits_{s\in S}\overline{u}_i(s)+\tfrac{\gamma}{2}-\max\nolimits_{s\in S}\overline{u}_i(s)=\tfrac{\gamma}{2}.
\end{aligned}$$

The contradiction $0<\gamma<\frac{\gamma}{2}$ concludes the proof. ∎

Corollary 1. *If in the normal-form game $\Gamma=\{N,\,S_1,...,S_n,\,u_1,...,u_n\}$ exist a Nash equilibrium $s^*=(s_1^*,...,s_n^*)$ and a quasi-Nash equilibrium $\overline{s}_i^*=(\overline{s}_1^*,\overline{s}_2^*,...,\overline{s}_n^*)\in S$ in normal-form game $\overline{\Gamma}=\{N,\,S_1,...,S_n,\,\overline{u}_1,...,\overline{u}_n\}$, then*

$$\forall i\in N\quad |u_i(s^*)-\overline{u}_i(\overline{s}^*)|\leq\mu.$$

The basic problem that remains when defining a quasi-Nash equilibrium is to find out the situations in which it is possible to approximate the discontinuous payoff function with the correct continuous function as close as possible.

One possibility is to look at functions of an immediate type, such as w-continuous ([5]). Let (X,d) be a metric space with a distance d, $D(f)\subset X$ the domain of mapping f and $f:D(f)\to X$.

Definition 8 ([5]). *A mapping f is said to be w-continuous, $w>0$, at point $x_0\in D(f)$ if*

$$\forall\varepsilon>0\quad\exists\delta>0\quad\forall x\in D(f):\quad d(x_0,x)<\delta\Rightarrow d(f(x_0),f(x))<\varepsilon+w.$$

A mapping f is said to be w-continuous if it is w-continuous at every point of $D(f)$. For $w = 0$ we get the usual definition of a continuous mapping. The Dirichlet function $f : \mathbf{R} \to \{-1, 1\}$, $f(x) = \begin{cases} 1, & x \in \mathbf{Q}, \\ -1, & x \in \mathbf{I}, \end{cases}$, $x \in \mathbf{R}$, is 2-continuous ($\mathbf{Q}$ is the set of rational numbers, $\mathbf{I}$ is the set of irrational numbers). This function is 2-continuous at every point of the set $\mathbf{R}$, but the definition allows us to view it also as w'-continuous for any $w' > 2$. The constant w may not be the best possible (smallest) one. Very often, especially in economic applications (see [6, 7]), there is known only a rough upper estimation for the "jump". Exactly the constant w includes uncertainty about the deviation of a function from continuity.

If $D(f)$ is a compact set, then a w-continuous mapping f has a continuous approximation g such that $\forall x \in D(f) : \quad d(f(x), g(x)) \leq 2w'$ where $w' > w$ ([5]). If $f : [a, b] \to \mathbf{R}$ is a uniformly w-continuous mapping, then for every $w' > w$ there exists a continuous approximation g such that $\forall x \in [a, b] : \quad d(f(x), g(x)) \leq \frac{w'}{2}$ ([5]).

Definition 9 ([5]). *A mapping $f : D(f) \to X$ is said to be uniformly w-continuous, $w > 0$, if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in D(f) : \quad d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon + w.$$

Returning back to the normal-form game Γ with w-continuous payoff mapping, we know that it can be approximated by a continuous function depending on the parameter w, but the approximation function should be concave or quasi-concave. If this is possible, then a quasi-Nash equilibrium exists.

4 Examples and conclusions

Example 2. We consider an example from [8]. We consider a two player normal-form game $\Gamma = \{[0, 1], [0, 1], u_1, u_2\}$, where $u_2(x_1, x_2) = 0$ and

$$u_1(x_1, x_2) = \begin{cases} 1, \, x_1 \in [0, 1[, x_2 = 0, \\ 0, \, x_1 \in [0, 1[, x_2 \neq 0, \\ 2, \quad x_1 = 1, x_2 = 0, \\ 1, \quad x_1 = 1, x_2 \neq 0. \end{cases}$$

The graph of the function u_1 is given in Fig.1. a). In this case, it is possible to say that u_1 is 2-continuous since only at the point $(1, 0)$ there is a jump of 2 and elsewhere it is 1. Theoretically, a better continuous approximation than $\mu = 1$ cannot be obtained. This means that, for example, the function $g_1(x_1, x_2) = 1$ is suitable - it is continuous and concave. Unfortunately, in the game $\overline{\Gamma} = \{[0, 1], [0, 1], g_1, u_2\}$ all strategy pairs (x_1, x_2) are Nash equilibria and such a solution does not make sense.

Other better approximations are possible. It is possible to approximate u_1 by the plane $\overline{u}_1 = \frac{1}{2}(1 + x_1 - x_2)$, $x_1, x_2 \in [0, 1]$ (see Fig.1. b)). This is also an approximation with $\mu = 1$, also continuous and concave function. In this case, the best response for player 1 is $x_1^* = 1$, because for any choice x_2 of player 2, the payoff function of player 1 is increasing, reaching the highest value at the endpoint of the strategy set. The resulting quasi-Nash equilibrium $(1, x_2)$, $x_2 \in [0, 1]$, coincides with the Nash equilibrium in the original game.

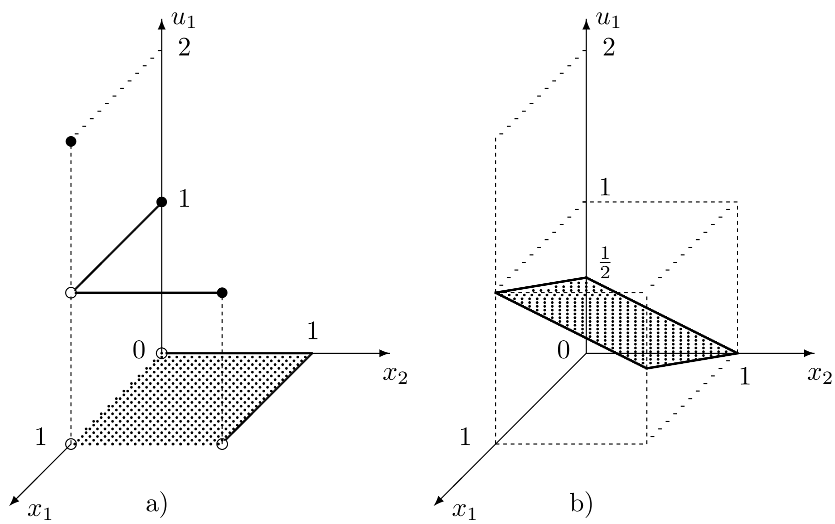

Fig.1. a) the graph of discontinuous payoff function u_1.
b) the graph of approximation of u_1 with continuous function.

■

Example 3. In the Cournot duopoly (Example 1), we have considered the players' payoffs as concave parabolas.

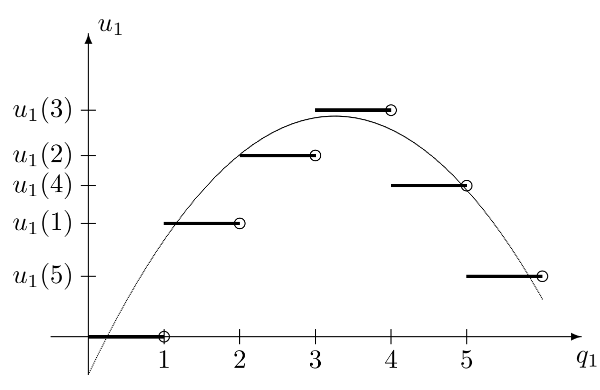

Fig.2. The graph of discontinuous payoff function and its approximation by a continuous function.

If we assume that the product can be sold only when a whole number of products have been produced, then the payoff function of producer 1 could look like Fig. 2.

Parabola $\overline{u}_1(q_1) = aq_1 + bq_1^2 + c$, $a < 0$, is the function in the class of real-valued functions that possess the concavity and continuity. Similarly, the 2-dimensional case could be approximated by a paraboloid or a general case, for example, by function $\overline{u}_1(q_1, ..., q_n) = a_1q_1^2 + a_2q_2^2 + ... + a_nq_n^2$. ■

The aim of the research is to develop an approximation that would give such quasi-Nash equilibria that coincide with the Nash equilibria of the base game, if they exist, and possibly give a reasonable solution in those cases where there is no Nash equilibrium.

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