Multiple viewpoint analogical proportions and their weighting*

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Abstract. Analogical proportions are statements of the form a is to b as c is to d. They are supposed to satisfy some postulates such as reflexivity, symmetry, stability under central permutation, or unicity of d when a, b, c are given. A simple Boolean model that satisfy all these postulates has been proposed fifteen years ago. In this model, analogical proportions are also transitive, which may be found more debatable. This short paper discusses how these postulates or properties (except reflexivity and symmetry) could be challenged, in agreement with a focused version of the Boolean model. By focused version we mean that the evaluation of an analogical proportion heavily depends on the set of attributes that should be considered. By varying this set, some properties such as the unicity of d given a, b, and c, or the transitivity may be lost. This type of observation also calls for a weighting of the importance of attributes or groups of attributes. It is argued that Sugeno integrals are of interest for this task.

Keywords: Analogical proportion · Postulates · Sugeno integral.

1 Introduction

Analogical proportions are statements of the form "a is to b as c is to d" (or "a is to b what c is to d") relating four items a, b, c, d of the same kind. This format is fairly liberal and open to various interpretations.

A traditional view of analogical proportions, which dates back to Aristotle [2], makes a parallel with numerical, arithmetical or geometrical, proportions (Aristotle may have been influenced by mathematical works on numerical proportions known in his time). This naturally leads to postulates such as symmetry and stability under central permutation. Such postulates are widely accepted for analogical proportions [6, 12], and apparently make sense for analogical statements such as, e.g., "the cow is to the calf as the mare is to the foal" (where the sentence still makes sense if we exchange 'calf' and 'mare'. However, the central permutation postulate may be debatable in other cases, for instance, in a sentence such that "wine is to the French what beer is to the English" where the exchange of 'French' and 'beer' is a bit awkward; note that in this latter case two universes, described by distinct sets of features, are involved: beverages and peoples.

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About fifteen years ago, a simple Boolean model has been proposed for analogical proportions in [17, 18]. This model obeys the basic postulates. This model has strong consequences such as the transitivity of analogical proportions, or the unicity of d given a, b and c. Such properties may be found also debatable.

However, we show in this paper that these properties can be lost if we adopt a multiple viewpoint in assessing analogical proportions. Indeed the evaluation of an analogical proportion strongly depends on what subset of attributes we focus on, and varying the focus modify the evaluations.

The paper is organized as follows. Section 2 discusses postulates, their universal consequences, and their consequences inside the Boolean model. Section 3 shows the interest of exploiting the attributes on which the proportion focuses for explanation purposes, or for making clear why some properties may no longer hold. This leads us in Section 5 to suggest that it maybe useful to weight attributes in the evaluation of analogical proportions. The appropriateness of Sugeno integrals is advocated.

2 Postulates, Boolean model and properties

This section first recalls the postulates traditionally associated with analogical proportions [6] and discusses them. The Boolean model is also recalled with its additional properties, before considering a weaker system of postulates.

2.1 Background

Postulates and their consequences The three properties that are usually postulated for an analogical proportion "a is to b as c is to d", denoted $a : b :: c : d$, are the three following ones, whatever the nature of the items a, b, c and d considered.

(P1) $a : b :: a : b$ (*reflexivity*);
(P2) $a : b :: c : d \Rightarrow c : d :: a : b$ (*symmetry*);
(P3) $a : b :: c : d \Rightarrow a : c :: b : d$ (*central permutation*).

The repeated, alternate application of symmetry and central permutation shows that $a : b :: c : d$ can be rewritten in 7 other equivalent forms: $c : d :: a : b$, $c : a :: d : b$, $d : b :: c : a$, $d : c :: b : a$, $b : a :: d : c$, $b : d :: a : c$, and $a : c :: b : d$. This shows that these two postulates have universal consequences like:

- $a : b :: c : d \Rightarrow b : a :: d : c$ (*internal reversal*);
- $a : b :: c : d \Rightarrow d : b :: c : a$ (*extreme permutation*);
- $a : b :: c : d \Rightarrow d : c :: b : a$ (*complete reversal*).

Moreover, reflexivity and central permutation yields

- $a : a :: b : b$ (*identity*).

Boolean model Viewing a, b, c, d as Boolean variables with value in $\mathbb{B} = \{0, 1\}$, various equivalent Boolean formulas satisfy the postulates of an analogical proportion. One of them making explicit that "a differs from b as c differs from d (and vice-versa)" [18] is:

$$a : b :: c : d = ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d)) \quad (1)$$

It is easy to check that this formula is only true for the 6 valuations in Table 1. As shown in [19], this set of 6 valuations is the *minimal* Boolean model obeying the 3 postulates of analogy.

0 : 0 :: 0 : 0
1 : 1 :: 1 : 1
0 : 1 :: 0 : 1
1 : 0 :: 1 : 0
0 : 0 :: 1 : 1
1 : 1 :: 0 : 0

Table 1. Boolean valuations that make the analogical proportion true

Besides, it can be seen on this table that 1 and 0 play symmetrical roles, which makes the definition *code-independent*. This is formally expressed with the negation operator $\neg$ as:

$$a : b :: c : d \Rightarrow \neg a : \neg b :: \neg c : \neg d.$$

Moreover, in this setting, it can be checked that the analogical proportion is *transitive*:

$$(a : b :: c : d) \wedge (c : d :: e : f) \Rightarrow a : b :: e : f.$$

However it is clear that code-independence and transitivity are not consequences of the postulates, that is they may fail in other models.

Lastly, in the Boolean model the following holds:

$a : b :: c : d, a : b :: c : x \Rightarrow x = d$ (*unicity of the solution when it exists).*

This property is taken as a postulate by some authors [12]. However, the solution may not exist. Indeed $1 : 0 :: 0 : x$ and $0 : 1 :: 1 : x$ have no solution (since we cannot give a value to x in $\mathbb{B}$ and obtain one of the valuations in Table 1).

2.2 Weaker postulates

If we abandon postulate $(P3)$ (stability under central permutation) because it would be felt too strong, a *weaker* set of postulates for analogical proportions [1] may be:

(P1) $a : b :: a : b$ (*reflexivity*);
(P2) $a : b :: c : d \Rightarrow c : d :: a : b$ (*symmetry*);
(P4) $a : b :: c : d \Rightarrow b : a :: d : c$ (*internal reversal*).

where $(P3)$ is replaced by $(P4)$ (*internal reversal*), one of the joint consequences of $(P3)$ with symmetry. Then, *complete reversal* ($a : b :: c : d \Rightarrow d : c :: b : a$) is still a consequence of this weaker set of postulates. According to postulates $(P1)$-$(P2)$-$(P4)$ $a : b :: c : d$ can be written only under 4 equivalent forms rather than 8: $a : b :: c : d$, $c : d :: a : b$, $d : c :: b : a$, and $b : a :: d : c$. Clearly a strong analogical proportion (in the sense of $(P1)$-$(P2)$-$(P3)$) is also a weak proportion. Despite one might be tempted to have $a : a :: b : b$ (*identity*), this is no longer deducible from the postulates $(P1)$-$(P2)$-$(P4)$. The minimal Boolean model of this latter set of postulates corresponds to the 4 first lines in Table 1, which are the models of formula $(a \equiv c) \wedge (b \equiv d)$.

As discussed in [13, 1], a proportion $a : b :: c : d$ is often understood in computational linguistics (see e.g., [5, 7, 15]), as: *for some binary relation* R, $R(a, b) \wedge R(c, d)$ *holds*, where R is some relation supposed to be non trivial. Such a definition perfectly fits with postulates $(P1)$-$(P2)$-$(P4)$, and *internal reversal* reads $R^{-1}(b, a) \wedge R^{-1}(d, c)$ *holds*. Note that the example of an analogical statement that does not accept *central permutation*, "wine is to the French as beer is to the English", tolerates *internal reversal*, namely "the French is to wine as the English is to beer" (here R is "to be the traditional alcoholic beverage of" and R^{-1} refers to "what people drinks as alcoholic beverage".

A more constrained form of analogical proportion, which may be viewed as a special case of the relational view above, can be written as $x : f(x) :: y : f(y)$, for some function f; see [11] for an extensive study of such proportions. For proper choices of f, central permutation $(P3)$ still makes sense; see [3] for some linkage with the Boolean model.

3 Analogies with variable focus

We first recall the multi-attribute modeling of analogical proportions, before explaining how some properties may fail when one focuses on different subsets of properties.

3.1 Multi-attribute analogical proportions

To deal with items represented by Boolean vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, it is straightforward to extend the definition of analogical proportion from $\mathbb{B}$ to $\mathbb{B}^n$ componentwise:

$$\vec{a} : \vec{b} :: \vec{c} : \vec{d} =_{def} \forall i \in [1, n],\ a_i : b_i :: c_i : d_i$$

Because of its definition componentwise, all postulates from the previous sections are still valid for multi-view analogical proportions. For instance, $\vec{a} : \vec{b} :: \vec{a} : \vec{b}$. This definition makes clear on what features the analogical proportion is based. For instance, using the 5 attributes $mammal$, $bovine$, $equine$, $adult$, $young$, Table 2 emphasizes why "the cow is to the calf as the mare is to the foal".

	$mammal$	$marsupial$	$bovine$	$equine$	$adult$	$young$
cow	1	0	1	0	1	0
calf	1	0	1	0	0	1
mare	1	0	0	1	1	0
foal	1	0	0	1	0	1

Table 2. A Boolean validation of $cow : calf :: mare : foal$

Indeed $a : b :: c : d = cow : calf :: mare : foal$ can rigorously be considered as a valid analogy since we recognize patterns of Table 1, vertically, in Table 2. It makes also clear that "among the mammals, the cow and the calf are respectively adult and young bovines, while the mare and the foal are respectively adult and young equines".

We see that such *explanatory* statement can be built systematically on the basis of the different valuation patterns observed vertically. Namely, the statement is of the form "among the X's, a and b are respectively Y U and Z U, while c and d are respectively Y V and Z V", where

- X refers to features common to a, b, c, d; it corresponds to pattern $1 : 1 :: 1 : 1$;
- Y refers to features common to a and c; it corresponds to pattern $1 : 0 :: 1 : 0$;
- Z refers to features common to b and d; it corresponds to pattern $0 : 1 :: 0 : 1$;
- U refers to features common to a and b; it corresponds to pattern $1 : 1 :: 0 : 0$;
- V refers to features common to c and d; it corresponds to pattern $0 : 0 :: 1 : 1$.

It would be also possible to take the pattern $0 : 0 :: 0 : 0$, by starting the statement with "among the X's that are not T ..." where T refers to features absent in the four items (in the previous example, $T = marsupial$). In case we have several attributes with the same pattern, we have to combine the corresponding features. It is also worth pointing out that the explanation format given above corresponds exactly to a definition of analogical proportion making clear what a, b, c, d have in common, what a, b have in common and is proper to them, what c, d have in common and is proper to them, how a and b differ and how the same difference holds between c and d; see, e.g., [18].

Interestingly enough the four items a, b, c, d have distinct representations only if the analogical proportion presents at least one attribute with a pattern of the form $s : t :: s : t$ and one attribute with a pattern of the form $s : s :: t : t$, where $s \neq t$ and $s, t \in \mathbb{B}$. These two patterns are exchanged by central permutation. This indicates that the model of an analogical proportion without the *central permutation* postulate cannot support analogical proportions such as "the cow is to the calf as the mare is to the foal", where the analogical proportion relies on features that differentiate the pairs (a, b) and (c, d), together with features that differentiate the pairs (a, c) and (b, d).

3.2 Multiple viewpoint analogical proportions

We have just seen that the Boolean model of analogical proportion between items described by vectors of Boolean attribute values depends on a subset of attributes that supports the analogical proportion, and offers a basis for explaining it.

If we change the subset of attributes considered for evaluating an analogical proportion, this may obviously have an impact on the result. Very often items are described by multiple attributes but only a subset of them are usually considered in analogical proportions between items. The choice of this subset may have an impact on some of the properties we have previously discussed, as shown now.

Unicity We first show that the unicity of $\vec{d}$, given $\vec{a}$, $\vec{b}$, and $\vec{c}$, making an analogical proportion $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$, is lost when we can vary the focus of the analogical proportion.

We start with an abstract example given in Table 3. Let us denote by $(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S$ the fact that the analogical proportion holds componentwise for all attributes $i \in S$, S being the focus for the analogical proportion. Then it can be easily seen that we have both $(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S$ and $(\vec{a} : \vec{b} :: \vec{c} : \vec{d'})_{S'}$, with $S = \{i_1, i_2, i_3\}$ and $S' = \{i_3, i_4, i_5\}$ respectively. In all cases, as expected, $d_i = d'_i$ for $i \in S \cap S'$; moreover $d_j \neq d'_j$ for $i \notin S \cap S'$ if S and S' are the largest subsets of attributes where $(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S$ and $(\vec{a} : \vec{b} :: \vec{c} : \vec{d'})_{S'}$ hold respectively.

	i_1	i_2	i_3	i_4	i_5
$\vec{a}$	1	1	0	0	1
$\vec{b}$	1	1	1	0	1
$\vec{c}$	1	0	0	0	0
$\vec{d}$	1	0	1	1	1
$\vec{d'}$	0	1	1	0	0

Table 3. Failure of unicity

Table 4 exhibits a concrete example. Indeed the chickadee is to the albatross as the mouse is to the elephant inasmuch the chickadee and the albatross are respectively small and big birds as the mouse and the elephant are respectively small and big mammals. Similarly, the chickadee is to the albatross as the mouse is to the whale inasmuch the chickadee and the albatross are respectively small terrestrial and big marine birds as the mouse and the whale are respectively small terrestrial and big marine mammals. Observe also that, more simply, the chickadee is to the albatross as the mouse is to the whale inasmuch the chickadee and the albatross are respectively small and big birds as the mouse and the whale are respectively small and big mammals. Indeed in this example, we already have the two solutions, $elephant$ and $whale$, for making an analogical proportion with $chickadee$, $albatross$ and $mouse$, if we restrict ourselves to attributes $bird$, $mammal$, $small$ and big, since the two vectors $\vec{d}$ and $\vec{d'}$ are equal on these attributes.

	$bird$	$mammal$	$small$	big	$terrestrial$	$marine$
$\vec{a} = chickadee$	1	0	1	0	1	0
$\vec{b} = albatros$	1	0	0	1	0	1
$\vec{c} = mouse$	0	1	1	0	1	0
$\vec{d} = elephant$	0	1	0	1	1	0
$\vec{d'} = whale$	0	1	0	1	0	1

Table 4. Example of failure of unicity

Transitivity Although transitivity holds in the Boolean variables model, some readers might object that analogical proportions may not be transitive. This remark is valid, and this becomes more apparent when dealing with multiple focus. Specifically $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ may hold with respect to some attributes and $\vec{c} : \vec{d} :: \vec{e} : \vec{f}$ may hold with respect to a different subset of attributes leading to a failure of transitivity, as in the following abstract example.

Assume $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$ can be described in terms of 4 Boolean attributes i_1, i_2, i_3, i_4, and $\vec{a} = (1, 1, 0, 0)$, $\vec{b} = (1, 1, 1, 0)$, $\vec{c} = (1, 0, 0, 0)$, $\vec{d} = (1, 0, 1, 1)$, $\vec{e} = (0, 1, 1, 0)$, and $\vec{f} = (0, 1, 1, 1)$. It can be easily checked that $(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_{\{i_1,i_2,i_3\}}$ holds as well as $(\vec{c} : \vec{d} :: \vec{e} : \vec{f})_{\{i_1,i_2,i_4\}}$, while $(\vec{a} : \vec{b} :: \vec{e} : \vec{f})_{\{i_1,i_2,i_3,i_4\}}$ does not hold. Still, it can be observed that here transitivity is preserved if we restrict ourselves to the subset of attributes common to the two analogical proportions, namely $S = \{i_1, i_2\} = \{i_1, i_2, i_3\} \cap \{i_1, i_2, i_4\}$.

	i_1	i_2	i_3	i_4
$\vec{a}$	1	1	0	0
$\vec{b}$	1	1	1	0
$\vec{c}$	1	0	0	0
$\vec{d}$	1	0	1	1
$\vec{e}$	0	1	1	0
$\vec{f}$	0	1	1	1

Table 5. Failure of transitivity

Analogical proportions have been extended to nominal and to numerical attribute values (normalized in the unit interval) [10]. These extensions preserve transitivity [19] and unicity. The analysis of the failure of these properties in case of analogies with variable focus still applies to these extensions.

4 Examplifying the failures of transitivity and unicity

In this section, we illustrate how transitivity or unicity can fail within a real-world context.[1] We analyze a dataset sourced from the US Congress [8], offering a comprehensive snapshot of voting behavior across 16 diverse topics. This 'Voting' dataset comprises 435 records, each representing a distinct voting profile. Because a vote can only be Yes or No, this is a typical Boolean dataset. The 16 features are related to a vote Yes or No on diverse topics such as:

"handicapped_infants","water_project_cost_sharing","physician_fee_freeze", "adoption_of_the_budget_resolution","el_salvador_aid", etc.

Furthermore, this 'Voting' dataset categorizes voters into either Democrats or Republicans. The dataset serves as a standard benchmark for classification tasks. Research has demonstrated the effectiveness of the analogical proportion-based classifier [4], achieving an accuracy rate of 94.7%.

In this context, an analogy $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ can be read as "a voter with profile $\vec{a}$ behave w.r.t. a voter with profile $\vec{b}$ exactly as a voter with profile $\vec{c}$ behave w.r.t. a voter with profile $\vec{d}$". Focusing on different subsets of attributes, we can then compute the number of analogies we can observe in the dataset. Let us describe an experiment. We randomly sample 10 voting profiles in the dataset. We then build $10 * 9/2 = 45$ distinct pairs, and then $45 * 44 = 1980$ pairs of pairs, candidates for building analogical proportions with four distinct profiles. We choose a focus $focus_0 = [1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ indicating that the set S of features we are interested in is constituted with the 7 first features. Changing the focus to $focus_1 =$

[1] The whole Python code used in these experiments is freely available on GitHub repository https://github.com/gillesirit/multiViewAP.

$[1, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1]$, we get another number of analogies among the 1980 candidates. Obviously, changing the 10 initial vectors would lead to other values that can be checked on the GitHub repository.

4.1 Transitivity

Recall that transitivity for multiple viewpoints analogical proportions focusing on S is expressed as:

$$(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S \wedge (\vec{c} : \vec{d} :: \vec{e} : \vec{f})_S \implies (\vec{a} : \vec{b} :: \vec{e} : \vec{f})_S$$

As previously explained, what we want to observe is the fact that, in general, neither:

$$(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S \wedge (\vec{c} : \vec{d} :: \vec{e} : \vec{f})_{S'} \implies (\vec{a} : \vec{b} :: \vec{e} : \vec{f})_S$$

nor

$$(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_S \wedge (\vec{c} : \vec{d} :: \vec{e} : \vec{f})_{S'} \implies (\vec{a} : \vec{b} :: \vec{e} : \vec{f})_{S'}$$

but

$$(\vec{a} : \vec{b} :: \vec{c} : \vec{d})_{S \cap S'} \wedge (\vec{c} : \vec{d} :: \vec{e} : \vec{f})_{S \cap S'} \implies (\vec{a} : \vec{b} :: \vec{e} : \vec{f})_{S \cap S'}$$

So a pair of analogical proportions $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ and $\vec{c} : \vec{d} :: \vec{e} : \vec{f}$ constitutes a candidate analogical pattern to test transitivity. We then proceed as follows:

- We build the lists l_0 (resp. l_1) of all the analogies with focus $focus_0$ (resp. $focus_1$). In general the two lists are different. If $focus_1$ is a subset of $focus_0$ (we focus on less attributes), then necessarily l_0 is a subset of l_1.
- We check that when a candidate analogical pattern $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ and $\vec{c} : \vec{d} :: \vec{e} : \vec{f}$ belong both to list l_0 (resp. l_1), then $\vec{a} : \vec{b} :: \vec{e} : \vec{f}$ belongs to l_0 (resp. l_1) by virtue of transitivity.
- We compute the list of candidate analogical patterns with $\vec{a} : \vec{b} :: \vec{c} : \vec{d} \in l_0$ and $\vec{c} : \vec{d} :: \vec{e} : \vec{f} \in l_1$. Then we check if $\vec{a} : \vec{b} :: \vec{e} : \vec{f} \in l_0$ or not, and if $a : b :: e : f \in l_1$ or not.

Figure 1 is a screenshot of a run. As expected, by observing this figure, we see that when focusing on a unique list, the number of candidate analogical patterns is equal to the number of observed transitivities: 134 for l_0 and 64 for l_1. But when we mix, we get 68 candidate analogical patterns, but only 29 lead to transitivity (i.e., a valid analogy in l_0 or l_1), see line 'focus01' in Figure 1. As expected, transitivity no longer always holds when we adopt a multiple viewpoint for analogical proportions.

```
dataset size: 10 dimension: 16
Total number of pairs: 45 – number of pairs of pairs: 1980
Total number of analogies found with focus0 = [1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] : 40
Total number of analogies found with focus1 = [1, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1] : 30
With focus0, number of candidates: 134 number of successful transitivity: 134
With focus1, number of candidates: 64 number of successful transitivity: 64
With focus01, number of candidates: 68 number of successful transitivity: 29
```

Fig. 1. Transitivity test

4.2 Unicity

To investigate unicity, we proceed as follows:

- We use the same 10 initial profiles (initially randomly chosen in the full dataset).
- We use the same focus sets $focus_0$ and $focus_1$ as the ones used for transitivity testing.
- We take 3 distinct profiles $\vec{a}, \vec{b}, \vec{c}$ (among the 10 initial profiles) such that the equation $\vec{a} : \vec{b} :: \vec{c} : \vec{x}$ is solvable w.r.t. $focus_0$ and also w.r.t. $focus_1$. Note that such profiles do not always exist (depending of the initial sample of 10 profiles). We get one such triplet in the example given in the GitHub.
- We solve the analogical equation $\vec{a} : \vec{b} :: \vec{c} : \vec{x}$, first focusing on $focus_0$ (resp. $focus_1$), getting solution sol_0 (resp. sol_1) and we may observe $sol_0 \neq sol_1$ in case of failure of unicity.
- Additionally, we count the number of profiles within the complete dataset of 435 profiles, coinciding with sol_0 (resp. sol_1) on attributes of $focus_0$ (resp. $focus_1$). It is worth noting that a profile thus 'matching' sol_0 (resp. sol_1) necessarily constitutes a solution to the equation $\vec{a} : \vec{b} :: \vec{c} : \vec{x}$, with a focus on $focus_0$ (resp. $focus_1$).

As illustrated in the screenshot provided in the GitHub repository, we identify 49 profiles 'matching' sol_0 and 44 'matching' sol_1 with $sol_0 \neq sol_1$. Unicity is definitely not a valid property when we adopt a multiple viewpoint for analogical proportions. The GitHub repository, containing the code, also includes complete screenshots summarizing the entire computation process, accompanied by detailed comments for clarity.

In addition to exhibiting deficiencies in transitivity and unicity, the experiment conducted on the 'Voting' dataset suggests that analogical proportions may play a role in the analysis of a dataset. Specifically, in cases where transitivity is preserved, we obtain equivalence classes of pairs [20]. These equivalence classes depend on the attributes in the considered focus. A change of focus breaking transitivity may be revealing of some facts of interest. Exploring possible applications of these observations in dataset analysis is a topic for further research.

5 Weighting attributes in analogies

The idea of weighting attributes in analogical proportion is not new. It can be already found in [16]. In classification problems, the application of analogical inference relies on the use of triplets $\vec{a}, \vec{b}, \vec{c}$ whose classes are known, that form an analogical proportion with $\vec{d}$ whose class $cl(\vec{d})$ is unknown. Then the fact that $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ holds[2] suggest that $cl(\vec{d})$ may be solution of the analogical equation on classes, $cl(\vec{a}) : cl(\vec{b}) :: cl(\vec{c}) : x$ (if the solution exists), see, e.g., [4].

But it is not sure that the attributes used for describing the data in the dataset, are all relevant or useful for classification. However the fact that $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ holds is judged on *all* attributes used for describing the items. So, if we suspect that some attributes are not relevant, it is tempting to discount them completely, or at least to diminish their

[2] This becomes a matter of degree in case of numerical attribute values [4].

impact on the global evaluation. In the following, we advocate that Sugeno integrals may be appropriate for this task. We begin with a brief review of Sugeno integrals.

5.1 Sugeno integral: a brief refresher

We consider a finite set of n attributes, $\mathcal{A} = \{1, \cdots, n\}$. A capacity (or fuzzy measure) is a set function $\mu : 2^{\mathcal{C}} \to L$ such that $\mu(\emptyset) = 0$, $\mu(\mathcal{A}) = 1$ and $\forall A, B \subseteq \mathcal{A}$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. Here the capacity μ representing a weighting system on the attributes, where not only single attributes, but also groups thereof, can be weighted. Then a Sugeno integral of $x = (x_1, \cdots, x_n)$ based on capacity μ is given by the following expression.

$$S_\mu(x) = \max_{A \subseteq \mathcal{A}} \min(\mu(A), \min_{i \in A} x_i).$$

It can be shown that

$$\min_{x_i : i \in \mathcal{C}} x_i \leq S_\mu(x) \leq \max_{x_i : i \in \mathcal{C}} x_i.$$

Here x_i corresponds to the evaluation of an analogical proportion on the vectorial component i: $a_i : b_i :: c_i : d_i$. we have $\forall i, x_i \in [0, 1]$, since in the general case we do not only consider Boolean analogical proportion, but also analogical proportions on numerical attributes (normalized in $[0, 1]$) [10].

Moreover when μ is a possibility measure Π (resp. a necessity measure N)[3] then the Sugeno integral reduces to a weighted maximum of the form $\max_i \min(x_i, \lambda_i)$ (resp. minimum of the form $\min_i \max(x_i, 1 - \lambda_i)$), where $\forall i, \lambda_i \in [0, 1]$ and $\max_i \lambda_i = 1$. The more important attribute i, the larger λ_i. Obviously, when $\forall i, \lambda_i = 1$, the maximum and minimum operations are retrieved.

5.2 Sugeno integrals in analogies

As explained in the introduction of this section, the implicit aggregation of the evaluation x_i of the analogical proportions $a_i : b_i :: c_i : d_i$ is the minimum since the analogical proportion should hold for each i. When attributes are numerical, the evaluation of an analogical proportion becomes a matter of degree [10]. The minimum operation sanctions the attribute(s) where the analogical proportion holds the least. In case some attributes are not considered as very relevant, this minimum aggregation may be replaced by a weighted minimum aggregation. If an attribute i is of importance λ_i[4], even if the analogical proportion does not hold at all on i, it's as if the evaluation is at least $1 - \lambda_i$.

In the previous section, we have seen that analogical proportions between four items may have different focuses depending on what subset of attributes is considered. This calls for an aggregation where each focus is a focal element of the capacity which is

[3] Possibility measures are max-decomposable for the union, namely $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$, and necessity measures are min-decomposable for the intersection, namely $N(A \cap B) = \min(N(A), N(B))$ [9].

[4] Such a level of importance may be assessed on the basis of a relevance index inspired by the comparison of pairs in analogical proportions [14].

used. Namely a capacity μ can be reconstructed from a function containing the minimum information to determine μ, called its qualitative Möbius transform, defined by

$$\mu_{\#}(T) = \begin{cases} \mu(T) & \text{if } \mu(T) > \max_{i \in T} \ \mu(T \setminus \{i\}) \\ 0 & \text{else} \end{cases}$$

We have $\forall A \subseteq \mathcal{A}, \quad \mu(A) = \max_{T \subseteq A} \mu_{\#}(T)$.

The sets T for which $\mu_{\#}(T) > 0$ are called the focal sets of μ, and $\mu_{\#}(T)$ are their weights of importance. This offers a basis for obtaining a global evaluation combining the different focuses of an analogical proportion. This may be of interest for aggregating multiple viewpoints in analogical proportion evaluation.

6 Conclusion

This short paper has made several (modest) contributions. First, it has provided a discussion of postulates and properties underlying analogical proportions. Second, it has been pointed out that analogical proportions between Boolean vectors can be explained (or justified) in terms of the attributes involved. Third, the failure of properties such as unicity of solution of analogies, or transitivity, can be explained by the existence of multiple focuses corresponding to different subsets of attributes to be considered in the evaluation of the analogical proportions. Lastly, the weighting of the importance of attributes involved has been discussed.

It may come as a surprise to see a paper mainly on analogical proportions in a session in memory of Michio Sugeno. Yet the range of his research interests was very broad, and included linguistics. Moreover, the first author clearly recalls a discussion, at a conference, about analogical proportions with Michio Sugeno and Tomohiro Takagi who was using them in a recommendation system [21] at that time.

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