Analysis of the φ-index of inclusion restricted to a set of indexes

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Abstract. The φ-index of inclusion has proven to be a suitable generalization of the inclusion in the fuzzy setting. In this paper, the properties of the φ-index of inclusion, when its definition is restricted to a subset of indexes, are analyzed. The theoretical results obtained in this work are necessary in order to develop fuzzy inference systems based on the φ-index of inclusion.

Keywords: Inclusion measure, Fuzzy inclusion, Fuzzy sets, φ-index of inclusion.

1 Introduction

The φ-index of inclusion was presented originally in [9] as a novel approach to model the inclusion between fuzzy sets and it was extended to general L-fuzzy sets in [6]. The main difference with respect to the existing approaches in the literature is that the inclusion between fuzzy sets is represented by mappings, instead of by values in the lattice of truth degrees.

Since this seminal approach [9], many different results have been obtained to support its use as a good generalization in the fuzzy setting of the classical inclusion of sets. For example, it was proved in [5], that the φ-index of inclusion satisfies almost all the axioms proposed by Fan-Xie-Pei [3], Sinha-Dougherty [10] and Kitainik [4], after a convenient rewrite according to the functional structure of Ω. In [8], it was shown that the φ-index of inclusion and the φ-index of contradiction [1] can be used to define a square of opposition, in the line of the Aristotelian one. Another interesting result, reached in [7], shows that the mapping that defines φ-index of inclusion can be used in a modus ponnens inference which is optimal with respect to the set of modus ponens defined from residuated pairs.

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The results obtained so far for the φ-index of inclusion, motivate its use for developing fuzzy inference systems. With this goal in mind, we have realized that it is necessary to analyze the properties of the φ-index of inclusion when its definition is restricted to a subset of indexes. This paper provides the first steps in this direction.

The structure of the paper is organized as follows. In Section 2 we recall the notion of f-inclusion and the definition of the φ-index of inclusion. Then, in Section 3 we analyze the properties of the φ-index of inclusion when the set of possible f-inclusions are restricted to a certain set. Finally, in Section 4 we describe future works and conclusions.

2 The notion of f-inclusion

Fuzzy sets are defined as usual, that is a *fuzzy set* A is defined by means of its membership function $A\colon \mathcal{U} \to [0,1]$, where $\mathcal{U}$ is a set called universe. The standard operations between sets are extended as follows: given two fuzzy sets A and B, we define the fuzzy sets

- $(A \cup B)(u) = \max\{A(u), B(u)\}$;
- $(A \cap B)(u) = \min\{A(u), B(u)\}$;
- $A^c(u) = 1 - A(u)$.

Let us recall now some preliminary notions needed to define the φ-index of inclusion. The first one is the set of indexes of inclusion, which plays an essential role in the definition of the φ-index of inclusion [9].

Definition 1. *The* set of indexes of inclusion*, denoted by Ω, consists of all the deflationary and monotonically increasing mapping; that is, any mapping $f\colon [0,1] \to [0,1]$ such that*

- *$f(x) \leq x$, for all $x \in [0,1]$ and*
- *$x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in [0,1]$.*

Another necessary notion to define the φ-index of inclusion is the definition of f-inclusion which is recalled below.

Definition 2. *Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that A* is f-included *in B, denoted by $A \subseteq_f B$, if and only if the inequality $f(A(u)) \leq B(u)$ holds for all $u \in \mathcal{U}$.*

Note that mappings $f \in \Omega$ can be identified with different degrees of inclusion, since each f-inclusion determines a different degree of restriction between fuzzy sets; that is, the greater $f \in \Omega$ the more restrictive is the f-inclusion. Taking into account the complete lattice structure of $(\Omega, \leq, \wedge, \vee)$ with respect the point-wise ordering (i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in [0,1]$ with the identity function, $id(x) = x$ for all $x \in [0,1]$, as the top element and the mapping $\perp(x) = 0$ for all $x \in [0,1]$ as the bottom element), we can provide the following definition.

Definition 3. *Let A and B be two fuzzy sets, the* φ-index of inclusion*, denoted by $Inc(A, B)$, is defined as*

$$Inc(A, B) = \sup\{f \in \Omega \mid A \subseteq_f B\}.$$

Due to the lack of space, we cannot provide a complete description of the properties and technical details of the φ-index of inclusion here. For such a reason, for a more detailed information related to the previous definitions, we refer the reader to [6, 9]. Below we enumerate the main properties of the φ-index of inclusion. Firstly, it can be proved that the supremum operator appearing in Definition 3 is actually a maximum. In addition, the set of f-inclusions between A and B can be represented by means of $Inc(A, B)$, as the following proposition shows.

Proposition 1 ([9]). *Let A and B be two fuzzy sets and $f \in \Omega$, then $A \subseteq_f B$ if and only if $f \leq Inc(A, B)$.*

The following theorem summarizes the main properties of Inc that are aligned to the axiomatic approaches of Fan-Xie-Pei [3], Sinha-Dougherty [10] and Kitainik [4].

Theorem 1 ([6]). *Let A, B and C be fuzzy sets,*

1. *(Full inclusion) $Inc(A, B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
2. *(Null inclusion) $Inc(A, B) = \perp$ if and only if there is a set of elements in the universe $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ such that $A(u_i) = 1$ for all $i \in I$ and $\bigwedge_{i \in I} B(u_i) = 0$.*
3. *(Pseudo transitivity) $Inc(B, C) \circ Inc(A, B) \leq Inc(A, C)$; where $\circ$ denotes the standard composition of functions.*
4. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(C, A) \leq Inc(B, A)$.*
5. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(A, B) \leq Inc(A, C)$.*
6. *(Transformation Invariance) Let $T \colon \mathcal{U} \to \mathcal{U}$ be a bijective mapping[1] on $\mathcal{U}$, then $Inc(A, B) = Inc(T(A), T(B))$.*
7. *(Relationship with intersection) $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.*
8. *(Relationship with union) $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.*

The axiom related to the complement, that is, the contraposition rule, is the only one which is not directly satisfied by $Inc(A, B)$. In general, we have that $Inc(A, B) \neq Inc(B^c, A^c)$, but this property is also captured by the φ-index of inclusion by means of a natural transformation (an integral) that turns the function $Inc(A, B)$ into a value in $[0, 1]$, as it was proved in [6].

Proposition 2 ([5]). *Let (f, g) be an isotone Galois connection[2], then $A \subseteq_f B$ if and only if $B^c \subseteq_{1-g(1-x)} A^c$.*

[1] Usually, in approaches related to measures of inclusion, e.g., [10], bijective mappings in $\mathcal{U}$ are called transformations.

[2] A pair of functions (f, g) forms an isotone Galois Connection if the following holds: $f(x) \leq y$ if and only of $x \leq g(y)$, for all $x, y \in [0, 1]$.

f-index of inclusion restricted to a complete join-sub-semilattice of Ω

The standard operators in fuzzy logic (as t-norms) define subsets of indexes of inclusion satisfying that, under such restrictions, the φ-index of inclusion coincides with the standard measure of inclusion [7]. This fact motivates the consideration of subsets in Ω for the definition of the φ-index of inclusion.

Definition 4 ([7]). *Let A and B be two fuzzy sets and Θ be a complete join-sub-semilattice of Ω; i.e., Θ is closed under arbitrary suprema and contains $\perp$ and id. Then, the* φ-index of inclusion restricted to Θ, *denoted by $Inc_\Theta(A,B)$, is defined as*

$$Inc_\Theta(A,B) = \sup\{f \in \Theta \mid A \subseteq_f B\}.$$

The following example makes use of the product t-norm to illustrate this fact. Note that the same conclusions are obtained when other t-norms, e.g., Gödel or Łukasiewicz, are considered.

Example 1. We define the set of indexes of inclusions by means of the product t-norm, $x *_P y = x \cdot y$ with $x, y \in [0,1]$, as the set $\Theta_P = \{\alpha *_P x \mid \alpha \in [0,1]\}$. Then, it is proven in [7, Theorem 3] that, given two fuzzy sets A and B, we have that

$$Inc_{\Theta_P}(A,B)(x) = \left(\bigwedge_{u \in \mathcal{U}} A(u) \to_P B(u) \right) *_P x$$

where $\to_P$ denotes the residuated implication of the product t-norm. As a result, the computation and meaning of $Inc_{\Theta_P}(A,B)$ is equivalent to:

$$\bigwedge_{u \in \mathcal{U}} A(u) \to_P B(u)$$

which is the standard measure of inclusion with respect to the product residuated implication.

3 Properties of Inc_Θ

The analysis of the operator Inc_Θ to represent a degree of inclusion between two fuzzy sets was not considered in [7], since the aim of that paper was oriented to provide a relationship of Inc with residuated implications. In this section, we will analyze the properties presented in Theorem 1, for the case $\Theta \neq \Omega$. For the sake of presentation, hereafter we assume that Θ represents a complete join-sub-semilattice of Ω; i.e., Θ is closed under arbitrary suprema and contains $\perp$ and id.

Let us begin with the following lemma, which will be used later on.

Lemma 1. *Let A and B be two fuzzy sets, then:*

$$Inc_\Theta(A,B) \leq Inc(A,B)$$

Proof. The proof comes directly from the definition of supremum and $\Theta \subset \Omega$. Specifically, we have that

$$\{f \in \Theta \mid A \subseteq_f B\} \subseteq \{f \in \Omega \mid A \subseteq_f B\}$$

and then

$$\sup\{f \in \Theta \mid A \subseteq_f B\} \leq \sup\{f \in \Omega \mid A \subseteq_f B\}$$

□

The full inclusion for Inc_Θ is represented by the greatest mapping in Θ, which is the identity function, as it happens for Inc. The following result shows that the full inclusion is equivalent to the Zadeh's inclusion.

Proposition 3. *Let A and B two fuzzy sets, then $Inc_\Theta(A, B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*

Proof. Since id is the supremum of Θ, $Inc_\Theta(A, B) = id$ is equivalent to say that $A \subseteq_{id} B$, which is equivalent to say that $A(u) \leq B(u)$ for all $u \in \mathcal{U}$. □

The null inclusion for Inc_Θ is represented by the least mapping in Θ, which is the function $\perp$, as it happens for Inc. However, the characterization of the null inclusion of Inc_Θ requires a different formulation than for Inc.

Proposition 4. *Let A and B two fuzzy sets, then $Inc_\Theta(A, B) = \perp$ if and only if, for each $f \in \Theta$ with $f \neq \perp$, we have that there exists an element $u \in \mathcal{U}$ such that $f(A(u)) > B(u)$.*

Proof. Since $A \subset_\perp B$, $Inc_\Theta(A, B) = \perp$ if and only if A is not f-included in B, for any $f \in \Theta$ with $f \neq \perp$, which is equivalent to say that there exists an element $u \in \mathcal{U}$ such that $f(A(u)) > B(u)$, for all $f \in \Theta$ with $f \neq \perp$. □

The pseudo-transitivity with respect to the composition between functions does not hold in general, as the following example shows.

Example 2. Let us consider the subset of indexes of inclusion given by $\Theta = \{id, f, \perp\}$ where f is defined by $f(x) = \frac{x}{2}$, for all $x \in [0, 1]$. Let A, B and C be three fuzzy sets on the universe $\mathcal{U} = \{u\}$ defined by $A(u) = 1$, $B(u) = \frac{1}{2}$ and $C(u) = \frac{1}{4}$. Then, it is easy to prove that

$$Inc_\Theta(A, B) = Inc_\Theta(B, C) = f \qquad \text{and} \qquad Inc_\Theta(A, C) = \perp.$$

Hence, we have that $Inc_\Theta(B, C) \circ Inc_\Theta(A, B)(x) = \frac{x}{4} \not\leq Inc_\Theta(A, C)(x) = 0$. □

The following result shows that if we impose that Θ is closed under composition of mappings, then the pseudo-transitivity holds.

Proposition 5. *Let A, B and C be three fuzzy sets and let us assume that for all $f, g \in \Theta$ we have $f \circ g \in \Theta$. Then, $Inc_\Theta(B, C) \circ Inc_\Theta(A, B) \leq Inc_\Theta(A, C)$.*

Proof. By Lemma 1 and Theorem 1, we have that $Inc_\Theta(B, C) \circ Inc_\Theta(A, B) \leq Inc(B, C) \circ Inc(A, B) \leq Inc(A, C)$. Therefore, by Proposition 1, we have that A is $Inc_\Theta(B, C) \circ Inc_\Theta(A, B)$-included in B.

By hypothesis, $Inc_\Theta(B, C) \circ Inc_\Theta(A, B) \in \Theta$, and then we have, by definition of Inc_Θ, that $Inc_\Theta(B, C) \circ Inc_\Theta(A, B) \leq Inc_\Theta(A, C)$. □

The following result shows that Inc_Θ is increasing on the second component and decreasing on the first one, as the monotonicity of Inc.

Proposition 6. *Let A, B and C be three fuzzy sets. Then:*

- *If $B(u) \leq C(u)$, for every $u \in \mathcal{U}$, then $Inc_\Theta(C, A) \leq Inc_\Theta(B, A)$.*
- *If $B(u) \leq C(u)$, for every $u \in \mathcal{U}$, then $Inc_\Theta(A, B) \leq Inc_\Theta(A, C)$.*

Proof. Let B and C be two fuzzy sets such that $B(u) \leq C(u)$, for all $u \in \mathcal{U}$. Let us prove the first item. Given $f \in \Theta$, by monotonicity of f we have that $f(B(u)) \leq f(C(u))$. Therefore, if $f(C(u)) \leq A(u)$, for every $u \in \mathcal{U}$, then $f(B(u)) \leq A(u)$. In other words, $C \subseteq_f A$ implies $B \subseteq_f A$, for all $f \in \Theta$. Consequently,

$$\{f \in \Theta \mid C \subseteq_f A\} \subseteq \{f \in \Theta \mid B \subseteq_f A\}$$

and then

$$Inc_\Theta(C, A) = \sup\{f \in \Theta \mid C \subseteq_f A\} \leq \sup\{f \in \Theta \mid B \subseteq_f A\} = Inc_\Theta(B, A)$$

The second item is proved similarly but noting that, in this case, $A \subseteq_f B$ implies $A \subseteq_f C$, for all $f \in \Theta$. □

The following result shows that the transformation invariance also holds for Inc_Θ.

Proposition 7. *Let A and B be two fuzzy sets and let $T \colon \mathcal{U} \to \mathcal{U}$ be a bijective mapping on $\mathcal{U}$, then $Inc_\Theta(A, B) = Inc_\Theta(T(A), T(B))$.*

Proof. Because of bijectivity of T, we have that $f(A(u)) \leq B(u)$, for any $u \in \mathcal{U}$, if and only if $f(A(T(u))) \leq B(T(u))$. Consequently, for each $f \in \Theta$ we have that $A \subseteq_f B$ if and only if $T(A) \subseteq_f T(B)$, which implies that $Inc_\Theta(A, B) = Inc_\Theta(T(A), T(B))$. □

The following result relates $Inc_\Theta(A, B \cap C)$ and $Inc_\Theta(A, B \cup C)$ to $Inc_\Theta(A, B)$ and $Inc_\Theta(A, C)$.

Proposition 8. *Let A and B be two fuzzy sets. Then*

- $Inc_\Theta(A, B \cap C) \geq Inc_\Theta(A, B) \wedge Inc_\Theta(A, C)$
- $Inc_\Theta(A \cup B, C) \geq Inc_\Theta(A, C) \wedge Inc_\Theta(B, C)$

Proof. The proof of this result is a direct consequence of Proposition 6. □

The following example shows that, in general, the equalities $Inc_\Theta(A, B \cap C) = Inc_\Theta(A, B) \wedge Inc_\Theta(A, C)$ and $Inc_\Theta(A \cup B, C) = Inc_\Theta(A, C) \wedge Inc_\Theta(B, C)$ does no hold in general.

Example 3. Let us consider the following set of f-indexes of inclusion $\Theta = \{id, f_1, f_2, f_3, f_4, f_5 \bot\}$, where $f_1(x) = \frac{x}{2}, f_2(x) = x^2$, $f_3(x) = x^4$, $f_4(x) = f_1(x) \vee f_2(x)$ and $f_5(x) = f_1(x) \vee f_3(x)$. Note that Θ is a join-sub-semilattice of Ω (note that $f_2 \geq f_3$ and then we do not need to consider $f_2 \vee f_3$). Let A, B and C be the fuzzy sets on the universe $\mathcal{U} = \{u_1, u_2\}$ given by

	A	B	C	$B \cap C$	$A \cup B$
u_1	$1/4$	$1/8$	$1/16$	$1/16$	$1/4$
u_2	$3/4$	$3/8$	$9/16$	$3/8$	$3/4$

The reader can easily check that $Inc_\Theta(A, B) = f_5 = f_1 \vee f_3$ and $Inc_\Theta(A, C) = f_2$. However, although A is $(f_1 \vee f_3) \wedge f_2$-included in $B \cap C$, we have that $(f_1 \vee f_3) \wedge f_2 \notin \Theta$, so $Inc(A, B \cap C) \neq Inc_\Theta(A, B) \wedge Inc_\Theta(A, C)$. A similar counter example for $Inc_\Theta(A \cup B, C) = Inc_\Theta(A, C) \wedge Inc_\Theta(B, C)$ can be constructed similarly. $\square$

The reader may think that the drawback presented in the previous example is because $(f_1 \vee f_3) \wedge f_2$ does not belong to Θ and the inclusion of such a mapping in Θ we solve the issue. However, this solution could not solve the problem, in general, since it is also necessary to guarantee that the structure of complete join-sub-semilattice in Θ is preserved.

Fortunately, we do not need to include more mappings in Θ to have such a property. Note firstly, that Θ always has a complete lattice structure thanks to it is closed under arbitrary suprema and it contains a minimum (the function $\bot$), as the following lemma states.

Lemma 2 ([2]). *Let $(L, \leq)$ be a complete join-semilattice with a minimum element, then $(L, \leq)$ is a complete lattice.*

Secondly, note that although the ordering in Θ coincides with the one in Ω, we could have a lattice structure in Θ that differs from the one of Ω; that is, the infimum and supremum operators of Ω could not coincide from those $\wedge_\Theta$ and $\vee_\Theta$ in Θ. The following example illustrate this fact.

Example 4. Let us consider Example 3. Note that

$$(f_1 \vee f_3) \wedge f_2(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 0.5 \\ \frac{x}{2} & \text{if } 0.5 \leq x < \frac{1}{\sqrt[3]{2}} \\ x^4 & \text{if } \frac{1}{\sqrt[3]{2}} \leq x \leq 1 \end{cases}$$

However, in the lattice structure of Θ we have $(f_1 \vee f_3) \wedge_\Theta f_2 = f_3$.

Therefore, the indexes of inclusion $Inc_\Theta(A, B \cap C)$ and $Inc_\Theta(A, B \cup C)$ can be obtained by means of the infimum in Θ, $Inc_\Theta(A, B)$ and $Inc_\Theta(A, C)$, as the following result shows.

Proposition 9. *Let A, B and C be fuzzy sets, and let $(\Theta, \leq, \wedge_\Theta, \vee_\Theta)$ be the complete lattice structure with respect to the ordering $\leq$ in Ω. Then*

- $Inc_\Theta(A, B \cap C) = Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)$.
- $Inc_\Theta(A \cup B, C) = Inc_\Theta(A, C) \wedge_\Theta Inc_\Theta(B, C)$.

Proof. Let us assume that $f = Inc_\Theta(A, B \cap C)$. By definition of infimum, we have that $g_1 \wedge_\Theta g_2 \leq g_1 \wedge g_2$ for all $g_1, g_2 \in \Theta$. Hence, by Proposition 8, we have directly that $f \geq Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)$, so let prove the opposite. Let us prove that A is $f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))$-included in B. Let $u \in \mathcal{U}$, then we have two possibilities for the value of $f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))(A(u))$, either

$$f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))(A(u)) = f(A(u)) \leq \min\{B(u), C(u)\} \leq B(u)$$

or

$$f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))(A(u)) = Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)(A(u)) \leq B(u)$$

So in both cases we have that $f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))(A(u)) \leq B(u)$ and, as a result, we can say that A is $f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C))$ included in B. Since $Inc(A, B)$ is the greatest index of inclusion of A in B, necessarily

$$f \vee (Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)) \leq Inc_\Theta(A, B)$$

As a result, we have that $f \leq Inc_\Theta(A, B)$. Similarly, we reach that $f \leq Inc_\Theta(A, C)$ and therefore, $f \leq Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)$, as we wanted to show.

The proof for $Inc_\Theta(A \cup B, C) = Inc_\Theta(A, C) \wedge_\Theta Inc_\Theta(B, C)$ follows similarly. □

The last result of this section imposes conditions on Θ to have the contraposition law in terms of Proposition 2.

Proposition 10. *Let A and B be two fuzzy sets, let $n(x) = 1 - x$ and let us assume that, for every $f \in \Theta$:*

- *there exists $\overline{f}\colon [0, 1] \to [0, 1]$ such that $(f, \overline{f})$ is an isotone Galois connection;*
- *$n \circ \overline{f} \circ n \in \Theta$.*

Then,

$$Inc_\Theta(B^c, A^c) = n \circ \overline{Inc_\Theta(A, B)} \circ n$$

where $\overline{Inc_\Theta(A, B)}$ is the only mapping such that $(Inc_\Theta(A, B), \overline{Inc_\Theta(A, B)})$ forms an isotone Galois connection.

Proof. It is a direct consequence of Proposition 2.

4 Conclusions and Future Work

In this paper, we have shown that we can define the φ-index of inclusion restricted to a certain set of f-indexes of inclusion Θ; named Inc_Θ. We have proven that most of the properties required by the axiomatic approach of Sinha-Dougherty [10] hold for Inc_{Θ}. The only problematic cases are the properties related to the Null inclusion, Transitivity relation (with respect to composition of functions) and Contrapositive law. These three drawbacks can be solved easily by requiring some inner structure in Θ; e.g, transitivity can be solved by requiring that Θ is closed by composition of mappings.

Establishing procedures to construct suitable subsets Θ of f-indexes of inclusion to fullfil the Sinha-Dougherty axioms is one of our future works. Our future work has also more applied oriented lines. We pretend to construct optimal subsets of f-indexes of inclusion for computational requirements (e.g., to simplify the computation of Inc_{Θ}); we pretend to develop a fuzzy inference system based on the generalized modus ponens that defines the φ-index of inclusion; and we pretend to provide a semantics based on the φ-index of inclusion in the context of fuzzy description logic.

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