From default to analogical and paralogical reasoning. Logics of pairs and their multiple-valued extensions*

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Abstract. The paper pursues the study of consequence relations associated with logical proportions. The latter are special expressions that combine conjunctively two equivalences between comparison indicators pertaining to two pairs (a, b) and (c, d) of Boolean variables. Analogical proportions, namely statements of the form a is to b as c is to d, are a special case of logical proportions in the Boolean case. There are three kinds of consequence relations in this setting. One is known for a long time and linked to default reasoning, another has been introduced last year and deals with the gain or lost of properties. The third one, associated with a logical proportion named paralogy (which states that what a and b have in common, c and d have it also and vice-versa) is studied in this paper together with its interplay with analogical proportions. Moreover, the paper presents multiple-valued extensions of the consequence relations.

Keywords: analogical proportion · consequence relation · logical proportion.

1 Introduction

Recently, a logic of ordered pairs of Boolean variables has been introduced and described [10] [11]. It is based on a relation of consequence logic $\models$ between two ordered pairs of vectors of Boolean variables $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ that states that if a property is gained when going from $\vec{a}$ to $\vec{b}$, it is also gained when going from $\vec{c}$ to $\vec{d}$, and that if a property is lost when going from $\vec{c}$ to $\vec{d}$, it was already lost when going from $\vec{a}$ to $\vec{b}$, as soon as $(\vec{a}, \vec{b}) \models (\vec{c}, \vec{d})$.

The possibility of defining such a logical consequence relation relies on a transitivity property of some logical proportions. Logical proportions [7] are Boolean logic expressions that state a conjunction of two equivalences between comparison indicators pertaining to two pairs (a, b) and (c, d) of Boolean variables. These comparison indicators refer to (positive or negative) similarity (the two variables are true, or are false, together), or to dissimilarity (one variable is true while the other is false). Among logical proportions, a small number of them are reflexive, symmetrical and transitive, and thus define equivalence relations between pairs. From these equivalence relations, it is then possible to define logical consequence relation, such that equivalence is retrieved when the corresponding logical consequence relation holds in both directions.

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Among the few logical proportions which define equivalence proportions, there are analogical proportions "a is to b as c is to d", which reads "a differs from b as c differs from d, and conversely, b differs from a as d differs from c". It gives birth to the consequence logic relation between ordered pairs preserving the gain of property described at the beginning of this introduction. There exists another family of logical proportions that state equivalence relations between two default rules by asserting that the two rules have the same examples and the same counter-examples [2]. The associated consequence relation is at the basis of nonmonotonic reasoning [3][1].

There is a last logical proportion stating an equivalence between pairs, named *paralogy* [7], which expresses that what a and b have in common (positively, or negatively), c and d have it also and vice-versa. This proportion can be related to analogical proportion, but its associated consequence relation, studied in this paper, is defined between pairs which are no longer ordered.

After recalling the necessary background in Section 2, we then study paralogical reasoning and its interplay with the logic of ordered pairs associated with analogical proportion in Section 3. We present multiple-valued extensions of the consequence relations introduced, in Section 4; thus, one can express, for instance, that if the satisfaction of a property is increased when going from $\vec{a}$ to $\vec{b}$, it increases at least as much when going from $\vec{c}$ to $\vec{d}$.

2 Background

We first revisit the concept of logical proportions and highlight the ones associated with equivalence relations between pairs. Then we take the example of the equivalence between default rules and its applications to non monotonic reasoning, before considering analogical proportions and the associated logic of ordered pairs.

2.1 Logical proportions

Logical proportions [7] are quaternary connectors expressing relations between pairs in propositional logic. They are built as follows. In Boolean logic, there are four indicators for comparing two variables a and b:

- Two indicators express *similarity*, either *positively* as $a \wedge b$ (which is true if both a and b are true), or *negatively* as $\neg a \wedge \neg b$ (which is true if a and b are both false).
- The other two are indicators of *dissimilarity* $\neg a \wedge b$ (which is true if a is false and b is true) and $a \wedge \neg b$ (which is true if a is true and b is false).

Then a logical proportion $T(a, b, c, d)$ is the conjunction of two equivalences between an indicator for (a, b) and an indicator for (c, d). For instance, the expression $((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((a \wedge b) \equiv (c \wedge d))$ offers an example of a logical proportion, where the same dissimilarity operator and the same similarity operator are applied to both pairs.

It has been established [7] that there are 120 semantically distinct logical proportions. Because of the way they are built, all these proportions $T(a, b, c, d)$ share a remarkable property: They are true for exactly 6 patterns of $abcd$ values among $2^4 = 16$

candidate patterns. For instance, the above proportion is true for 0000, 1111, 1010, 0101, 0001, and 0100. The interested reader is referred to [7, 8] for in-depth studies of the different types of logical proportions.

Among all 120 logical proportions T[1], viewed as defining a relation between two pairs, only 6 logical proportions are reflexive ($T(a, b, a, b)$ is a tautology), symmetrical ($T(a, b, c, d) = T(c, d, a, b)$), and transitive ($T(a, b, c, d), T(c, d, e, f) \Rightarrow T(a, b, e, f)$) [7]. These are:

- the 4 following, so-called, *conditional* logical proportions:
  $C_1(a, b, c, d) = ((a \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge \neg b) \equiv (c \wedge \neg d))$;
  $C_2(a, b, c, d) = ((a \wedge b) \equiv (c \wedge d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$;
  $C_3(a, b, c, d) = ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge \neg b) \equiv (\neg c \wedge \neg d))$;
  $C_4(a, b, c, d) = ((\neg a \wedge b) \equiv (\neg c \wedge d)) \wedge ((\neg a \wedge \neg b) \equiv (\neg c \wedge \neg d))$.
- the *analogical proportion*:
  $\mathrm{A}(a, b, c, d) = a : b :: c : d = ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$
- the so-called *paralogy*:
  $\mathrm{P}(a, b, c, d) = ((a \wedge b) \equiv (c \wedge d)) \wedge ((\neg a \wedge \neg b) \equiv (\neg c \wedge \neg d))$

In these 6 cases, the logical proportion defines an equivalence relation between pairs (a, b) and (c, d).

2.2 Conditional proportions and nonmonotonic reasoning

Let us first explain why the 4 proportions C_1, C_2, C_3, C_4 given in the previous subsection are called "conditional". It comes from the fact that these proportions express equivalences between conditional statements. Indeed, let us denote a rule "if a then b" by $b|a$, which is called a "conditional object" or a "conditional event". Now consider the proportion $C_1(a, b, c, d)$ (which was our example of a logical proportion):

$$((a \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge \neg b) \equiv (c \wedge \neg d)) \quad (1)$$

It expresses a semantic equivalence between the two rules "if a then b" and "if c then d", or if we prefer between $b|a$ and $d|c$, which can be also denoted $b|a :: d|c$ by combining the notation of conditional objects with that of the analogical proportion. Indeed

- $(a \wedge b) \equiv (c \wedge d)$ states they have the same examples since $a \wedge b$ and $c \wedge d$ are true in the same time;
- $(a \wedge \neg b) \equiv (c \wedge \neg d)$ states that they have the same counter-examples satisfying the condition parts of the rules and falsifying their conclusions ;
- observe that if a is false then c is false also (otherwise (1) would be false); thus if rule $b|a$ is not applicable, then rule $d|c$ is not applicable, and conversely.

Indeed a rule "if a then b" can be considered as a three-valued entity, either true, or false, or undefined, when the rule is not applicable (a false) [3]. The logical consequence relation between conditional objects $b|a \vDash_C d|c$ is defined as [2]:

$$b|a \vDash_C d|c \text{ iff } a \wedge b \vDash c \wedge d \text{ and } c \wedge \neg d \vDash a \wedge \neg b \quad (2)$$

[1] The interested reader may visit the website https://www.irit.fr/ Gilles.Richard/analogy/logic/ for discovering the different types of logical proportions, their truth valuations and properties.

which expresses that the examples of the first conditional object are examples of the second one, and the counter-examples of the second conditional object are counter-examples of the first one. This entailment is naturally associated with the conditional proportion $b|a :: d|c$, since

$$b|a :: d|c \Leftrightarrow b|a \vDash d|c \text{ and } d|c \vDash b|a.$$

The 4 conditional proportions are closely related together since it can be checked that $C_1(a,b,c,d) = b|a :: d|c$, $C_2(a,b,c,d) = a|b :: c|d$, $C_3(a,b,c,d) = a|\neg b :: c|\neg d$, and $C_4(a,b,c,d) = b|\neg a :: d|\neg c$ (indeed it can be checked that (1) is stable when changing b into $\neg b$ and d into $\neg d$).

A rule may have exceptions. That is, we can have at the same time the rule "if a then b" and a rule "if $(a \wedge c)$ then $\neg b$". The two conditional objects $b|a$ and $\neg b|(a \wedge c)$ do not lead to a contradiction in the presence of the facts a and c (unlike a modeling of rules by material implication), in the setting of a tri-valued logic where the conjunction $\&$ is defined by [3]:

$$b|a \;\&\; d|c \triangleq ((a \to b) \wedge (c \to d))|(a \vee c)$$

where $\to$ is the material implication ($a \to b \triangleq \neg a \vee b$) and with the following semantics: $val(b|a \;\&\; d|c) = \min(val(b|a), val(d|c))$ where the three truth values are ordered as follows: undefined $>$ true $>$ false.[2]

It can be shown that this quasi-conjunction '&' (that is its name) is associative. It expresses that the set constituted by the two rules "if a then b" and "if c then d" can be triggered if a or c is true, and in this case the triggered rule behaves like the material implication. This logic constitutes the simplest semantics [1] of the system P of non-monotonic inference of Kraus, Lehmann, and Magidor [4]. The reader can consult [3, 1] for more details. Nonmonotonic reasoning is here a two steps-process. First from a set of conditional events representing a set of default rules, we *deduce* a new conditional event whose condition part corresponds exactly to our knowledge of the current situation under consideration, and then - second step - we apply the new default rule thus inferred to the current situation.

2.3 Analogical proportion and the logic of pairs

The analogical proportion a is to b as c is to d, whose logical expression is

$a : b :: c : d = ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$

expresses that "a differs from b as c differs from d, and b differs from a as d differs from c", and is true only for the 6 valuations for $abcd$ given in Table 1. As can be seen in Table 1, there is no difference between a and b (resp. c and d) in the first 4 lines, while the differences are the same in the last two.

The analogical proportion is the only logical proportion obeying the following three postulates:

- Reflexivity : $T(a,b,a,b)$

[2] The negation is defined by $\neg(b|a) = (\neg b|a)$; $\neg(b|a)$ is undefined if and only if $b|a$ is.

a	b	c	d
0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

Table 1. Boolean valuations making $a : b :: c : d$ true

- Symmetry : $T(a, b, c, d) \Rightarrow T(c, d, a, b)$
- Stability under central permutation: $T(a, b, c, d) \Rightarrow T(a, c, b, d)$

Thus, Table 1 offers the minimal Boolean model assuming reflexivity and stability under central permutation. This model is symmetrical.

Analogical proportions can be extended componentwise to items represented by vectors such as $\vec{a} = (a_1, ..., a_n)$ defined on the same set of Boolean attributes, namely:

$$\vec{a} : \vec{b} :: \vec{c} : \vec{d} \quad \text{iff} \quad \forall i \in [1, n],\ a_i : b_i :: c_i : d_i.$$

Let us define

$Equ^0(\vec{a}, \vec{b}) = \{i \mid a_i = b_i = 0\}$,
$Equ^1(\vec{a}, \vec{b}) = \{i \mid a_i = b_i = 1\}$,
$Equ(\vec{a}, \vec{b}) = \{i \mid a_i = b_i\} = Equ^0(\vec{a}, \vec{b}) \cup Equ^1(\vec{a}, \vec{b})$,
and
$Dif^{10}(\vec{a}, \vec{b}) = \{i \mid a_i = 1, b_i = 0\}$,
$Dif^{01}(\vec{a}, \vec{b}) = \{i \mid a_i = 0, b_i = 1\}$,
$Dif(\vec{a}, \vec{b}) = \{i \mid a_i \neq b_i\} = Dif^{01}(\vec{a}, \vec{b}) \cup Dif^{10}(\vec{a}, \vec{b})$.

This allows us to state the following result:

$$\vec{a} : \vec{b} :: \vec{c} : \vec{d} \text{ if and only if } \begin{cases} Equ(\vec{a}, \vec{b}) = Equ(\vec{c}, \vec{d}) \\ Dif^{10}(\vec{a}, \vec{b}) = Dif^{10}(\vec{c}, \vec{d}) \\ Dif^{01}(\vec{a}, \vec{b}) = Dif^{01}(\vec{c}, \vec{d}). \end{cases} \tag{3}$$

Analogical proportions define equivalence classes of pairs of Boolean vectors; $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ states that the pairs $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are in the same equivalence class. Following property (3), Table 2 partitions the attributes involved in an analogical proportion into subparts: one where the vectors are equal inside pairs (perhaps in a different way according to the pair) and a subpart where the same change takes place inside each pair. Observe that the "All equal" subpart may be empty, without trivializing the analogical proportion (i.e., the four vectors remain all different), this is not the case for the two other subparts that cannot be empty.

Condition $Equ(\vec{a}, \vec{b}) = Equ(\vec{c}, \vec{d})$ ensures that $H(\vec{a}, \vec{b}) = H(\vec{c}, \vec{d})$, where H denotes the Hamming distance $H(\vec{a}, \vec{b}) = |\ \{i \ \mid a_i \neq b_i\}\ |$.

Analogical inference relies on the solving of analogical equations $a_i : b_i :: c_i : x_i$ for some unknown value x_i of some attribute i. The solution is unique when it exists, but it may not exist. Indeed $1 : 0 :: 0 : x$ and $0 : 1 :: 1 : x$ have no solution as analogical equations.

$items$	$All\ equal$	$Equality\ by\ pairs$	$Change\ (Dif)$
$\vec{a}$	1 0	1 0	1 0
$\vec{b}$	1 0	1 0	0 1
$\vec{c}$	1 0	0 1	1 0
$\vec{d}$	1 0	0 1	0 1

Table 2. The 3 subparts of analogical proportion and the associated valuations

In [10, 11] a comparative logic of ordered pairs has been proposed. The items to be compared are described by vectors of attribute values (here Boolean). When $a_i = 1$ (resp. $a_i = 0$) we understand it as item $\vec{a}$ has (resp. has not) feature / property i.

For associating a logical consequence relation $\vDash_A$ with analogical proportion, we follow the same procedure as in the case of conditional logical proportions for defining $\vDash_C$ (definition (2)) from the conditional proportion C_1 defined by (1). Namely a logical consequence relation $\vDash_A$ between pairs is defined as:

$$(\vec{a}, \vec{b}) \vDash_A (\vec{c}, \vec{d}) \Leftrightarrow \neg\vec{a} \wedge \vec{b} \vDash \neg\vec{c} \wedge \vec{d} \text{ and } \vec{c} \wedge \neg\vec{d} \vDash \vec{a} \wedge \neg\vec{b} \quad (4)$$

where logical connectives are extended to vectors componentwise:

$\vec{a} \wedge \vec{b} = (a_1 \wedge b_1, ..., a_n \wedge b_n)$; $\vec{a} \vee \vec{b} = (a_1 \vee b_1, ..., a_n \vee b_n)$; $\neg\vec{a} = (\neg a_1, ..., \neg a_n)$.

When we deal with pairs, the valuation $(a_i, b_i) = (0, 1)$ is understood as: when we go from $\vec{a}$ to $\vec{b}$, we acquire feature i (the value of the attribute becomes true). Thus the meaning of entailment (4) is as follows: features that are acquired when going from $\vec{a}$ to $\vec{b}$ remain acquired when going from $\vec{c}$ to $\vec{d}$. Moreover if when going from $\vec{c}$ to $\vec{d}$ a feature is lost, it was already the case when going from $\vec{a}$ to $\vec{b}$.[3]

It can be checked that we have the following equivalence:

$$(\vec{a}, \vec{b}) \vDash_A (\vec{c}, \vec{d}) \text{ and } (\vec{c}, \vec{d}) \vDash_A (\vec{a}, \vec{b}) \text{ iff } A(\vec{a}, \vec{b}, \vec{c}, \vec{d})$$

In [10, 11], an operation denoted $\vee\!\!\wedge$, mixing conjunction and disjunction has been defined:

$$(\vec{a}, \vec{b}) \vee\!\!\wedge (\vec{c}, \vec{d}) = (\vec{a} \vee \vec{c}, \vec{b} \wedge \vec{d}).$$

This conjunctive-like operation agrees with this semantics of acquired features. Indeed one can check that $(a_i \vee c_i, b_i \wedge d_i) = (0, 1)$ only if $(a_i, b_i) = (c_i, d_i) = (0, 1)$. But, if (a_i, b_i) or $(c_i, d_i) = (1, 0)$, $(a_i \vee c_i, b_i \wedge d_i) = (1, 0)$. This shows the conjunctive nature of $\vee\!\!\wedge$ with respect to the idea of acquired feature.

Dually, we can define a disjunctive-like operation between pairs:

$$(\vec{a}, \vec{b}) \wedge\!\!\vee (\vec{c}, \vec{d}) = (\vec{a} \wedge \vec{c}, \vec{b} \vee \vec{d})$$

Note that $(a_i \wedge c_i, b_i \vee d_i) = (1, 0)$ only if $(a_i, b_i) = (c_i, d_i) = (1, 0)$. By contrast, if (a_i, b_i) or $(c_i, d_i) = (0, 1)$, $(a_i \wedge c_i, b_i \vee d_i) = (0, 1)$. Thus, $\wedge\!\!\vee$ *cumulates* acquired features.

[3] The choice of definition (4), rather than $(\vec{a}, \vec{b}) \vDash (\vec{c}, \vec{d}) \Leftrightarrow \vec{a} \wedge \neg\vec{b} \vDash \vec{c} \wedge \neg\vec{d}$ and $\neg\vec{c} \wedge \vec{d} \vDash \neg\vec{a} \wedge \vec{b}$, is governed by the need here to privilege the gain of features rather than their loss. Indeed the alternative definition given in this footnote says that features that are lost when going from $\vec{a}$ to $\vec{b}$ remain lost when going from $\vec{c}$ to $\vec{d}$, and that if when going from $\vec{c}$ to $\vec{d}$ a feature is gained, it was already the case when going from $\vec{a}$ to $\vec{b}$.

There is a De Morgan duality between $\vee\!\wedge$ and $\wedge\!\vee$ with respect to the operation $\circlearrowleft$ defined by

$$\circlearrowleft(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$$

which has a negation flavor since if a feature is gained from $\vec{a}$ to $\vec{b}$, it is lost when going from $\vec{b}$ to $\vec{a}$. Namely

$$\circlearrowleft(\circlearrowleft(\vec{a}, \vec{b}) \vee\!\wedge \circlearrowleft(\vec{c}, \vec{d})) = (\vec{a}, \vec{b}) \wedge\!\vee (\vec{c}, \vec{d}).$$

It can be checked that $\vee\!\wedge$ behaves like a conjunction, and $\wedge\!\vee$ like a disjunction, in the sense that

$$(\vec{a}, \vec{b}) \vee\!\wedge (\vec{c}, \vec{d}) \vDash_A (\vec{a}, \vec{b}) \vDash_A (\vec{a}, \vec{b}) \wedge\!\vee (\vec{c}, \vec{d})$$

In [10, 11], the use of the disjunction $\wedge\!\vee$ that cumulate acquired properties, has been suggested for enlarging a set of ordered pairs representing pairs of items where desirable properties are acquired by the second item of the pair. This enlargement provides a better basis for creativity by analogical inference.

3 Paralogical reasoning

We now consider the paralogical proportion. We first recall its definition, its main properties, and its interplay with analogical proportion. Then we introduce a paralogical consequence relation and we study its behavior.

3.1 Paralogy

The paralogical proportion

$$\mathrm{P}(a, b, c, d) = ((a \wedge b) \equiv (c \wedge d)) \wedge ((\neg a \wedge \neg b) \equiv (\neg c \wedge \neg d))$$

expresses that what a and b have in common (positively or negatively), c and d have it also and vice-versa. $\mathrm{P}(a, b, c, d)$ is true only for the 6 valuations in Table 3. As can be seen,

- either both a and b are false (resp. true) and both c and d are false (resp. true) as well; this corresponds to the first two lines in the table; ‘
- or a and b have different truth values and c and d have different truth values as well.

a	b	c	d
0	0	0	0
1	1	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0

Table 3. Boolean valuations making $P(a, b, c, d)$ true

But these pairs are no longer ordered while they are ordered in the case of an analogical proportion. Thus, $P(a, b, c, d) = P(b, a, c, d) = P(a, b, d, c) = P(b, a, d, c)$.

As the analogical proportion, paralogy is code independent, this means that 1 and 0 play symmetrical roles and can be exchanged: namely $P(a, b, c, d) = P(\neg a, \neg b, \neg c, \neg d)$, as we have $A(a, b, c, d) = A(\neg a, \neg b, \neg c, \neg d)$. See Tables 1 and 3.

The paralogical proportion is the only logical proportion T obeying the following three postulates:

– $T(a, b, b, a)$ (reverse reflexivity);
– $T(a, b, c, d) = T(c, d, a, b)$ (symmetry);
– $T(a, b, c, d) = T(b, a, c, d)$ (stability under lateral permutation).

Moreover paralogy satisfies reflexivity $T(a, b, a, b)$ also. Note that Table 3 is the minimal Boolean model satisfying both $T(a, b, b, a)$ and $T(a, b, a, b)$. In the Boolean setting, these two properties are thus enough to have symmetry, stability under lateral permutation, and ... transitivity. If we agree that "black is to white as white is to black", we should adopt paralogy (instead of analogy)!

Note also that if $T(a, b, c, d)$ is an analogical proportion, and $T(c, d, e, f)$ is a paralogical proportion then $T(a, b, e, f)$ is an analogy or a paralogy. The same holds if $T(a, b, c, d)$ is a paralogy and $T(c, d, e, f)$ an analogy. This is due to the fact that equations such as $T(1, 1, x, y)$ or $T(0, 1, x', y')$ have solutions compatible with analogy ($x = y = 1$ or $x = y = 0$, and $x' = 0, y' = 1$) and solutions compatible with paralogy ($x = y = 1$ and $x' = 0, y' = 1$ or $x' = 1, y' = 0$).

Paralogical proportions can be extended to vectors describing items on the same set of Boolean attributes. Thus $P(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ is defined componentwise:

$$P(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \text{ iff } \forall i \in [1, n],\ P(a_i, b_i, c_i, d_i).$$

Yet, paralogical and analogical proportions are closely related. Indeed $P(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = A(\vec{a}, \vec{d}, \vec{c}, \vec{b})$[4]. This can be checked by permuting the order of the lines in Table 2, as done in Table 4 where we recognize, vertically, the valuations making true a paralogy.

$items$						
$\vec{a}$	1	0	1	0	1	0
$\vec{d}$	1	0	0	1	0	1
$\vec{c}$	1	0	0	1	1	0
$\vec{b}$	1	0	1	0	0	1

Table 4. Paralogy obtained by permuting the lines of the table of an analogical proportion

The truth table of the paralogy enables us to state the following result:

$$P(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \text{ if and only if } \begin{cases} Dif(\vec{a}, \vec{b}) = Dif(\vec{c}, \vec{d}) \\ Equ^1(\vec{a}, \vec{b}) = Equ^1(\vec{c}, \vec{d}) \\ Equ^0(\vec{a}, \vec{b}) = Equ^0(\vec{c}, \vec{d}) \end{cases} \quad (5)$$

This indicates that two pairs of vectors $\{\vec{a}, \vec{b}\}$ and $\{\vec{c}, \vec{d}\}$ (we use braces instead of parentheses since the pairs are not ordered) form a paralogy $P(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ if the values in each pair differ simultaneously, and if when they are equal, they are equal to the same value. Condition $Dif(\vec{a}, \vec{b}) = Dif(\vec{c}, \vec{d})$ ensures that $H(\vec{a}, \vec{b}) = H(\vec{c}, \vec{d})$, where H is the Hamming distance. As can be seen, (5) contrasts with the equation (3) describing analogical proportion between vectors where the directions of change inside pairs should be the same, while elsewhere the vectors may be equal in a different ways from

[4] Equivalently $A(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = P(\vec{a}, \vec{d}, \vec{c}, \vec{b}) = P(\vec{a}, \vec{d}, \vec{b}, \vec{c})$, and thus it yields another equivalent formula for an analogical proportion, namely $A(a, b, c, d) = (a \wedge d \equiv b \wedge c) \wedge (a \vee d \equiv b \vee c)$, since $(\neg a \wedge \neg d \equiv \neg b \wedge \neg c)$ is the same as $(a \vee d \equiv b \vee c)$.

a pair to the other. Here in (5) this is the converse, vectors in a pair should be equal in the same way, and dissimilarity is no longer oriented.

Thus the attributes with respect to a paralogy can be partitioned in three subsets as in Table 5. The paralogy is non trivial (i.e., the four vectors are different) when both the two subparts "non ordered Dif" and "ordered Dif" are non empty. The subpart "identity" may remain empty.

$items$	$non\ ordered\ Dif$	$ordered\ Dif$	$Identity$
$\vec{a}$	1 0	1 0	1 0
$\vec{b}$	0 1	0 1	1 0
$\vec{c}$	0 1	1 0	1 0
$\vec{d}$	1 0	0 1	1 0

Table 5. The 3 parts of paralogical proportion and the associated valuations

Paralogical inference amounts to solve equation of the form $P(a_i, b_i, c_i, x_i)$ for some feature i where the value x_i is unknown. The solution when it exists is unique. But, $P(1, 1, 0, x)$ and $P(0, 0, 1, x)$ have no paralogical solution. Indeed there is no d_i such that what is both common (positively or negatively) to a_i and b_i is also common to c_i and d_i, if c_i misses what is common to a_i and b_i.

Permuting $\vec{a}$ and $\vec{c}$ in Table 3 leads to the truth table of A for $\vec{c}, \vec{b}, \vec{a}, \vec{d}$, meaning that $P(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \equiv A(\vec{c}, \vec{b}, \vec{a}, \vec{d})$. An immediate consequence of this equivalence is the fact that the equation $A(a_i, b_i, c_i, x_i)$ is solvable iff the equation $P(c_i, b_i, a_i, x_i)$ is solvable and they have the same solution. For instance, $A(0, 0, 1, x)$ is solvable with $x = 1$ and this is equivalent to $P(1, 0, 0, x)$ is solvable with the same solution 1. $A(1, 0, 0, x)$ is not solvable just as $P(0, 0, 1, x)$. Thus paralogical inference is just another way of presenting analogical inference.

The paralogical proportion establishes a parallel between pairs. Two pairs, in the same class of equivalence, are "parallel" if inside each pair, they differ on the same features, and the four vectors all identical on the remaining features.

3.2 Paralogical consequence relation

Taking inspiration of analogical proportion, one can define a paralogical consequence relation $\vDash_P$ in the following way:

$$\{\vec{a}, \vec{b}\} \vDash_P \{\vec{c}, \vec{d}\} \Leftrightarrow \vec{a} \wedge \vec{b} \vDash \vec{c} \wedge \vec{d} \text{ and } \neg\vec{c} \wedge \neg\vec{d} \vDash \neg\vec{a} \wedge \neg\vec{b} \quad (6)$$

Equation (6) states that if a feature i is common to the items in $\{\vec{a}, \vec{b}\}$ it is also common for the items in $\{\vec{c}, \vec{d}\}$, and conversely if missing in both items in $\{\vec{c}, \vec{d}\}$, it is also missing in both items in $\{\vec{a}, \vec{b}\}$, i.e., $\forall i, (a_i, b_i) = (1, 1) \Rightarrow (c_i, d_i) = (1, 1)$ and $\forall i, (c_i, d_i) = (0, 0) \Rightarrow (a_i, b_i) = (0, 0)$. As shown in the left part of Table 6, if $(a_i, b_i) = (0, 1)$ or $(1, 0)$, then $(c_i, d_i) \neq (0, 0)$; if $(a_i, b_i) = (0, 0)$, (c_i, d_i) is unconstrained. Thus $\vDash_P$ privileges the preservation of common features that are present (but not those that are absent).[5]

[5] Obviously there exists a dual consequence relation that privileges the preservation of common features that are absent (but not those that are present): $\{\vec{a}, \vec{b}\} \vDash_{P''} \{\vec{c}, \vec{d}\} \Leftrightarrow \vec{c} \wedge \vec{d} \vDash \vec{a} \wedge \vec{b}$ and $\neg\vec{a} \wedge \neg\vec{b} \vDash \neg\vec{c} \wedge \neg\vec{d}$. We have then $\forall i, (a_i, b_i) = (0, 0) \Rightarrow (c_i, d_i) = (0, 0)$ and $\forall i, (c_i, d_i) = (1, 1) \Rightarrow (a_i, b_i) = (1, 1)$.

a	b	c	d	$(a,b) \vDash_P (c,d)$	$(c,d) \vDash_P (a,b)$	$P(a,b,c,d)$	$(a,b) \vDash_{P'} (c,d)$	$(c,d) \vDash_{P'} (a,b)$
0	0	0	0	1	1	1	1	1
0	0	0	1	1	0	0	0	1
0	0	1	0	1	0	0	0	1
0	0	1	1	1	0	0	0	0
0	1	0	0	0	1	0	1	0
0	1	0	1	1	1	1	1	1
0	1	1	0	1	1	1	1	1
0	1	1	1	1	0	0	1	0
1	0	0	0	0	1	0	1	0
1	0	0	1	1	1	1	1	1
1	0	1	0	1	1	1	1	1
1	0	1	1	1	0	0	1	0
1	1	0	0	0	1	0	0	0
1	1	0	1	0	1	0	0	1
1	1	1	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1
nb	of	values	'true'	**11**	**11**	**6**	**10**	**10**

Table 6. Entailments defined by (6) vs. paralogical proportion vs. entailments defined by (7)

It can be checked in Table 6 that we have the following equivalence:

$$(\vec{a},\vec{b}) \vDash_P (\vec{c},\vec{d}) \text{ and } (\vec{c},\vec{d}) \vDash_P (\vec{a},\vec{b}) \text{ iff } P(\vec{a},\vec{b},\vec{c},\vec{d}).$$

As in the case of $\vDash_A$, one may think of defining conjunctive or disjunctive combinations of ordered pairs, agreeing with the consequence relation (6) and making sense with respect to the interpretation of pairs. Natural componentwise definitions, including negation, satisfying a De Morgan duality, seem to be :

$$(\vec{a},\vec{b}) \wedge (\vec{c},\vec{d}) = (\vec{a} \wedge \vec{c}, \vec{b} \wedge \vec{d});$$

$$(\vec{a},\vec{b}) \vee (\vec{c},\vec{d}) = (\vec{a} \vee \vec{c}, \vec{b} \vee \vec{d}).$$

$$\neg(\vec{a},\vec{b}) = (\neg\vec{a}, \neg\vec{b})$$

Note that $\neg(\vec{a},\vec{b}) \neq \circlearrowleft (\vec{a},\vec{b}) = (\vec{b},\vec{a})$. With $\neg$ one changes absent features into present features and vice-versa, while with $\circlearrowleft$ one changes loss into gain and vice-versa. Clearly if $\vec{a}$ and $\vec{b}$ share a feature i, also shared by $\vec{c}$ and $\vec{d}$, then feature i is shared by $\vec{a} \wedge \vec{c}$ and $\vec{b} \wedge \vec{d}$.

It can be shown that

$$(\vec{a},\vec{b}) \wedge (\vec{c},\vec{d}) \vDash_P (\vec{a},\vec{b}) \vDash_P (\vec{a},\vec{b}) \vee (\vec{c},\vec{d}).$$

Indeed $(a_i \wedge c_i) \wedge (b_i \wedge d_i) \vDash (a_i \wedge b_i)$ and $a_i \wedge b_i \vDash (a_i \vee c_i) \wedge (b_i \vee d_i)$. Besides, the second condition defining $\vDash_P$, namely $\neg c_i \wedge \neg d_i \vDash \neg a_i \wedge \neg b_i$ is the same as $a_i \vee b_i \vDash c_i \vee d_i$, and thus we do have $a_i \vee b_i \vDash (a_i \vee c_i) \vee (b_i \vee d_i)$. Note that $\wedge$, $\vee$ ensure that if a feature is common to the vectors of two pairs, it is also common to the vectors of the combined pair.

Another option for defining a consequence relation from paralogy, now preserving both common features that are present and common features that are absent, are as follows:

$$(\vec{a},\vec{b}) \vDash_{P'} (\vec{c},\vec{d}) \Leftrightarrow \vec{a} \wedge \vec{b} \vDash \vec{c} \wedge \vec{d} \text{ and } \neg\vec{a} \wedge \neg\vec{b} \vDash \neg\vec{c} \wedge \neg\vec{d} \quad (7)$$

As can be seen in the right part of Table 6, $\{\vec{a},\vec{b}\}$, i.e., $\forall i, (a_i, b_i) = (1,1) \Rightarrow (c_i, d_i) = (1,1)$ and $\forall i, (a_i, b_i) = (0,0) \Rightarrow (c_i, d_i) = (0,0)$. Moreover we have, if $(c_i, d_i) = (0,0)$ (resp. $= (1,1)$) then $(a_i, b_i) \neq (1,1)$ (resp. $\neq (0,0)$).

It can be checked that we have the following equivalence also (see right part of Table 6):

$$(\vec{a},\vec{b}) \vDash_{P'} (\vec{c},\vec{d}) \text{ and } (\vec{c},\vec{d}) \vDash_{P'} (\vec{a},\vec{b}) \text{ iff } P(\vec{a},\vec{b},\vec{c},\vec{d})$$

However $\vDash_{P'}$ has no good behavior wrt to conjunction / disjunction of pairs, be them defined with $(\wedge, \vee)$ or with $(\vee\wedge, \wedge\vee)$.

4 Multiple-valued extensions

The definition of the analogical consequence relation $\vDash_A$ enforces that features gained when going from a to b should be also gained when going from c to d, and conversely for the features lost. More generally, features may be a matter of degree, and we may want to state that if a feature is increased when going from $\vec{a}$ to $\vec{b}$, it is also increased when going from $\vec{c}$ to $\vec{d}$. Let us assume that the a_i's, b_i's, c_i's, d_i's may be a matter of degree. In the following we drop the index i in the notation. Now the consequence relation defined by (4) can be extended to multiple-valued vectors by

$$(a,b) \vDash_{\mathcal{A}} (c,d) \text{ iff } \begin{cases} \min(1-a,b) \leq \min(1-c,d) \\ \min(1-c,d) \leq \min(1-a,b) \end{cases} \quad (8)$$

Here $a \in [0,1]$ (resp. $1-a$) estimates to what extent the feature is present (resp. absent), $\min$ is used as a conjunction (other choices may be considered), and $\leq$ is the usual entailment between graded properties. If both $(a,b) \vDash_{\mathcal{A}} (c,d)$ and $(c,d) \vDash_{\mathcal{A}} (a,b)$, then $c = a$ and $d = b$, or $c = 1-b$ and $d = 1-a$, in both cases we have $b - a = d - c$. This agrees with the two possible definitions of graded analogical proportions [9]:

$$a : b ::_L c : d = \begin{cases} 1- \mid (a-b) - (c-d) \mid, \\ \quad \text{if } a \geq b \text{ and } c \geq d, \text{ or } a \leq b \text{ and } c \leq d \\ 1 - \max(\mid a-b \mid, \mid c-d \mid), \\ \quad \text{if } a \leq b \text{ and } c \geq d, \text{ or } a \geq b \text{ and } c \leq d \end{cases} \quad (9)$$

$$a : b ::_C c : d = \min(1 - \mid \max(a,d) - \max(b,c) \mid, 1 - \mid \min(a,d) - \min(b,c) \mid) \quad (10)$$

which are equal to 1 when $c = a$ and $d = b$.

The paralogical consequence relation (6) can be extended similarly by

$$\{\vec{a},\vec{b}\} \vDash_{\mathcal{P}} \{\vec{c},\vec{d}\} \text{ iff } \begin{cases} \min(a,b) \leq \min(c,d) \\ \min(1-c,1-d) \leq \min(1-a,1-b) \end{cases} \quad (11)$$

Note that the second inequality is equivalent to $\max(a,b) \leq \max(c,d)$ just as $\neg\vec{c} \wedge \neg\vec{d} \vDash \neg\vec{a} \wedge \neg\vec{b}$ is equivalent to $\vec{a} \vee \vec{b} \vDash \vec{c} \vee \vec{d}$. If we have both $\{a,b\} \vDash_{\mathcal{P}} \{c,d\}$ and $\{c,d\} \vDash_{\mathcal{P}} \{a,b\}$, then either $c = a$ and $d = b$ or $d = a$ and $c = b$, which agrees with the following definition of the multiple-valued paralogy (which is equal to 1 in these cases):

$$P(a,b,c,d) = \min(1 - \mid \min(a,b) - \min(c,d) \mid, 1 - \mid \max(a,b) - \max(c,d) \mid).$$

5 Conclusion

This paper has provided an overview of the different logical proportions that can be associated with consequence relations. Beyond the fact that both analogical and paralogical proportions are reflexive, symmetrical and transitive, and thus define equivalence relation between pairs, these two proportions are very different. Indeed they obey different postulates, have different semantic interpretations; moreover paralogy, being reverse reflexive, includes valuations such as $(0,1,1,0)$ and $(1,0,0,1)$ in its truth table, two valuations known as maximizing analogical dissimilarity [5] and having maximal Kolmogoroff complexity [9], while the valuations making true the analogy have a smaller complexity. Even if a close parallel can be made between analogical and paralogical inference, the corresponding proportions seem to serve purposes that are quite different. Yet, while default and analogical reasoning have clear applications, the use of paralogical reasoning as another way of presenting analogical reasoning is an open question. The parallel between analogical and nonmonotonic reasoning modes, first discussed in [6], also calls for a more detailed study in the light of the consequence relations discussed in this paper; see [8] for a parallel in the inference mechanisms.

References

1. S. Benferhat, D. Dubois, and H. Prade. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intelligence*, 92(1-2):259–276, 1997.
2. B. De Finetti. La logique des probabilités. In *Congrès Int. de Philosophie Scientifique, IV. Induction et probabilité*, pages 31–39, Paris,, 1936. Hermann.
3. D. Dubois and H. Prade. Conditional objects as nonmonotonic consequence relationships. *IEEE Trans. on Syst., Man and Cyber.*, 24:1724–1740, 1994.
4. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.
5. L. Miclet, S. Bayoudh, and A. Delhay. Analogical dissimilarity: Definition, algorithms and two experiments in machine learning. *J. Artif. Intell. Res.*, 32:793–824, 2008.
6. H. Prade and G. Richard. Cataloguing/analogizing: A nonmonotonic view. *Int. J. Intell. Syst.*, 26(12):1176–1195, 2011.
7. H. Prade and G. Richard. From analogical proportion to logical proportions. *Logica Universalis*, 7(4):441–505, 2013.
8. H. Prade and G. Richard. Homogenous and heterogeneous logical proportions. *IfCoLog J. of Logics and their Applications*, 1(1):1–51, 2014.
9. H. Prade and G. Richard. Analogical proportions: From equality to inequality. *Int. J. of Approximate Reasoning*, 101:234 – 254, 2018.
10. H. Prade and G. Richard. First steps towards a logic of ordered pairs. In Z. Bouraoui and S. Vesic, editors, *Proc. 17th Europ. Conf. on Symbolic and Quantit. Approaches to Reasoning with Uncert. (ECSQARU'23), Arras, Sept. 19-22*, LNCS 14294, 198–209. Springer, 2023.
11. H. Prade and G. Richard. Analogical proportion-based induction: From classification to creativity. *J. of Applied Logics — IfCoLog J. of Logics and their Appl.*, 11:51–87, 2024.