Metric-Based Fuzzy Equivalence and Inequality Relations *

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Abstract. This work focuses on the studying of metric-based fuzzy equivalence and inequality relations within the framework of fuzzy logic. Namely, our emphasis lies in the examination of transitive fuzzy equivalence and inequality relations, governed by the Archimedean t-norm and t-conorm, respectively. Subsequently, we study aggregation operators, which preserve the properties of initial fuzzy equivalence and inequality relations, and illustrate their construction by employing the additive generators for different t-norms and t-conorms. Our research contributes to a deeper understanding of construction of metric-based fuzzy relations and their aggregation. This knowledge holds practical implications across diverse domains, enabling effective applications in real-world scenarios.

Keywords: Fuzzy equivalence relation · Fuzzy inequality relation · Aggregation operator.

1 Introduction

The concept of fuzzy equivalence relations was first introduced by Zadeh in 1974 [10]. Since then, extensive research has been conducted in this field, with notable contributions from various scholars such as [1], [7], [9]. The concept of fuzzy relations holds a pivotal role in addressing various challenges inherent in applied sciences, particularly within domains such as artificial intelligence, decision making, image processing, operations research, data mining and many others (see eg. [2], [4], [3], [5]).

In our work, we provide an overview of fundamental concepts related to t-norms, t-conorms, and involution. Following this introduction, we define fuzzy equivalence relations and demonstrate how they can be constructed using relevant t-norm additive generators and crisp metrics. Additionally, we define fuzzy inequality relations and illustrate their construction based on relevant t-conorms additive generators and crisp metrics.

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Furthermore, our research delves into aggregation of fuzzy equivalence and inequality relations. We offer a comprehensive examination of how these operators are constructed, providing proof of their formulation and detailing the process for deriving them for any equivalence and inequality. By elucidating the mechanisms underlying aggregation operators, we enhance our understanding of their significance in both theoretical frameworks and practical decision-making scenarios.

Overall, this work aims to provide a comprehensive understanding of metric-based fuzzy equivalence and inequality relations. Additionally, it explores aggregation operators designed to generate novel fuzzy equivalence and inequality relations from the initial fuzzy relations. The work provides also multiple examples which are aimed to be used in practical applications.

The paper is structured as follows. In Section 2, we commence by introducing the foundational principles of fuzzy logic that are used in the paper. Specifically, we introduce triangular norms, triangular conorms and subsequently explore their construction using additive generators. Section 3 is devoted to the fuzzy equivalence relation. Fuzzy inequality relations are explored in Section 4. Finally, in Section 5, we conclude the paper.

2 Preliminaries

We start with the definition of a t-norm which represents a generalized conjunction in fuzzy logic:

Definition 1. *[6] A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e. a function $T:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:*

(T1) Commutativity: $T(x,y) = T(y,x)$;
(T2) Associativity: $T(x,T(y,z)) = T(T(x,y),z)$;
(T3) Monotonicity: $T(x,y) \leq T(z,y)$, *when* $x \leq z$;
(T4) Boundary condition: $T(x,1) = x$.

Some of often used t-norms are mentioned below:

- $T_{\min}(x,y) = \min(x,y)$ minimum t-norm;
- $T_P(x,y) = x \cdot y$ product t-norm;
- $T_Ł(x,y) = \max(x+y-1,0)$ Łukasiewicz's t-norm;
- Hamacher's t-norm is defined as follows:

$$T_\lambda^H(x,y) = \begin{cases} 0, & \text{if } \lambda = x = y = 0 \\ T_D(x,y), & \text{if } \lambda = \infty \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)}, & \text{otherwise.} \end{cases}$$

A t-norm T is called Archimedean if and only if, for all pairs $(x,y) \in (0,1)^2$, there is $n \in N$ such that $T^{(n)}(x) = T(x,...,x) < y$. Product, Łukasiewicz and Hamacher t-norms are Archimedean while minimum t-norm is not.

Let us proceed with other fuzzy logic operators called an involution and a t-conorm.

Definition 2. *[6] An involution is a function $N : [0;1] \to [0;1]$, such that for all $x, y \in [0;1]$ it holds:*

(N1) $N(N(x)) = x$;
(N2) *If $x \leq y$, then $N(x) \geq N(y)$.*

Definition 3. *[6] A function $S : [0;1] \times [0;1] \to [0;1]$ is a t-conorm if it satisfies the following properties for all $x, y, z \in [0;1]$:*

(S1) Commutativity: $S(x,y) = S(y,x)$;
(S2) Monotonicity: $S(x,y) \leq S(z,y)$ *when* $x \leq z$;
(S3) Associativity: $S(x, S(y,z)) = S(S(x,y), z)$;
(S4) Boundary condition: $S(x,0) = x$.

Some examples of t-conorms that are dual to previously mentioned t-norms:

– $S_M(x,y) = \max(x,y)$ maximum t-conorm;
– $S_P(x,y) = x + y - x \cdot y$ product t-conorm;
– $S_L(x,y) = \min(x+y, 1)$ Łukasiewicz's t-conorm;
– Hamacher's t-conorm is defined as follows

$$S_\lambda^H(x,y) = \begin{cases} 1, & \text{if } \lambda = 0, x = y = 1 \\ S_D(x,y), & \text{if } \lambda = \infty \\ \frac{x+y-xy-(1-\lambda)xy}{1-(1-\lambda)xy}, & \text{otherwise.} \end{cases}$$

We proceed recalling an important tool for the construction and study of t-norms involving single argument real function (additive generator) and addition. Later we use the same tool for constructing fuzzy equivalence relations. Similar to the t-norm additive generator we will define additive generator of a t-conorm that later will be used in construction of fuzzy inequality relations.

Definition 4. *[6] Let $f : [x,y] \to [c,d]$ be a monotone function, where $[x,y]$ and $[c,d]$ are closed subintervals of the extended real line $[-\infty, \infty]$. The pseudo-inverse function $f^{(-1)} : [c,d] \to [x,y]$ of f is defined by*

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [x,y] \mid f(x) < y\}, & \text{if } f(x) < f(y), \\ \sup\{x \in [x,y] \mid f(x) > y\}, & \text{if } f(x) > f(y), \\ x, & \text{if } f(x) = f(y). \end{cases}$$

Definition 5. *[6] An additive generator $g : [0,1] \to [0,\infty]$ of a t-norm T is a strictly decreasing function which is also right-continuous in 0 and satisfies $g(1) = 0$, such that for all $(x,y) \in [0,1]^2$ we have*

$$g(x) + g(y) \in Ran(g) \cup [g(0), \infty],$$

$$T(x,y) = g^{(-1)}(g(x) + g(y)).$$

where $Ran(g)$ is the range of g, $g^{(-1)}$ - pseudo-inverse.

Examples of additive generators $g : [0,1] \to [0,\infty]$ and their pseudo-inverse $g^{(-1)} : [0,\infty] \to [0,1]$ functions of previously mentioned t-norms:

- Additive generator g_p and it's pseudo-inverse $g_p^{(-1)}$ of a product t-norm:

$$g_p(x) = \begin{cases} -\ln x, & x \neq 0 \\ \infty, & x = 0; \end{cases}$$

$$g_p^{(-1)}(y) = \begin{cases} e^{-y}, & y \neq \infty \\ 0, & y = \infty. \end{cases}$$

- Additive generator $g_{\text{Ł}}$ and it's pseudo-inverse $g_{\text{Ł}}^{(-1)}$ of a Łukasiewicz's t-norm:

$$g_{\text{Ł}}(x) = 1 - x;$$

$$g_{\text{Ł}}^{(-1)}(y) = \begin{cases} 1 - y, & y \in [0,1] \\ 0, & \text{otherwise} . \end{cases}$$

- Additive generator g_{H_λ} and it's pseudo-inverse $g_{H_\lambda}^{(-1)}$: of a Hamacher's t-norm:

$$g_{H_\lambda}(x,y) = \begin{cases} \frac{1-x}{x}, & if\ \lambda = 0 \\ \ln \frac{\lambda+(1-\lambda)x}{x}, & otherwise; \end{cases}$$

$$g_{H_\lambda}^{(-1)} = \begin{cases} \frac{1}{1+y}, & if\ \lambda = 0 \\ \frac{\lambda}{e^y - 1 + \lambda}, & otherwise. \end{cases}$$

Definition 6. *[6] An additive generator $g : [0,1] \to [0,\infty]$ of a t-conorm S is a strictly increasing function which is also left-continuous in 1 and satisfies $g(0) = 0$, such that for all $(x,y) \in [0,1]^2$ we have*

$$g(x) + g(y) \in Ran(g) \cup [g(1), \infty],$$

$$S(x,y) = g^{(-1)}(g(x) + g(y)).$$

where $Ran(g)$ is the range of g, $g^{(-1)}$ - pseudo-inverse.

Examples of additive generators $g : [0,1] \to [0,\infty]$ and their pseudo-inverse $g^{(-1)} : [0,\infty] \to [0,1]$ of previously mentioned t-conorms:

- Additive generator g_p and it's pseudo-inverse $g_p^{(-1)}$ of a product t-conorm

$$g_p(x) = \begin{cases} -\ln(1-x), & x \neq 1 \\ \infty, & x = 1; \end{cases}$$

$$g_p^{(-1)}(y) = \begin{cases} 1 - e^{-y}, & y \neq \infty \\ 1, & y = \infty. \end{cases}$$

- Additive generator $g_{\text{Ł}}$ and it's pseudo-inverse $g_{\text{Ł}}^{(-1)}$ of a Łukasiewicz's t-conorm

$$g_{\text{Ł}}(x) = x;$$

$$g_{\text{Ł}}^{(-1)}(y) = \begin{cases} y, & y \in [0,1] \\ 1, & \text{otherwise} . \end{cases}$$

– Additive generator g_{H_λ} and it's pseudo-inverse $g_{H_\lambda}^{(-1)}$ of a Hamacher's t-conorm

$$g_{H_\lambda}(x) = \begin{cases} \frac{x}{1-x}, \ if \ \lambda = 0 \\ \ln \frac{\lambda+(1-\lambda)(1-x)}{1-x}, \ otherwise; \end{cases}$$

$$g_{H_\lambda}^{(-1)}(y) = \begin{cases} \frac{x}{1+x}, \ if \ \lambda = 0 \\ 1 - \frac{\lambda}{e^x-1+\lambda}, \ otherwise. \end{cases}$$

3 Fuzzy equivalence relation

We continue with an overview of basic definitions and results on fuzzy equivalence relations.

Definition 7. *(see e.g. [7], [10]) A fuzzy binary relation R on a set M is a mapping $R : M \times M \to [0,1]$.*

Definition 8. *(see e.g. [9]) A fuzzy binary relation E on a set M is called a fuzzy equivalence relation with respect to a t-norm T (or T-equivalence), if and only if the following three axioms are fulfilled for all $x, y, z \in M$:*

1. *$E(x,x) = 1$ reflexivity;*
2. *$E(x,y) = E(y,x)$ symmetry;*
3. *$T(E(x,y), E(y,z)) \le E(x,z)$ T-transitivity.*

The following result establishes principles of construction of fuzzy equivalence relations using pseudo-metrics.

Theorem 1. *[1] Let T be a continuous Archimedean t-norm with an additive generator g. For any pseudo-metric d, the mapping*

$$E_d(x,y) = g^{(-1)}(\min(d(x,y), g(0)))$$

is a T-equivalence.

Let us consider 3 examples of fuzzy equivalences that are constructed using additive generators of previously mentioned t-norms:

Example 1. [T-equivalence for product t-norm]

$$E_P(x,y) = e^{-d(x,y)}.$$

Example 2. [T-equivalence for Łukasiewicz's t-norm]

$$E_L(x,y) = \max(1 - d(x,y), 0).$$

Example 3. [T-equivalence for Hamacher's t-norm]

$$E_{H_\lambda}(x,y) = \begin{cases} \frac{1}{1+d(x,y)}, \ if \ \lambda = 0 \\ \frac{\lambda}{e^{d(x,y)}-1+\lambda}, \ if \ \lambda \in]0;\infty]. \end{cases}$$

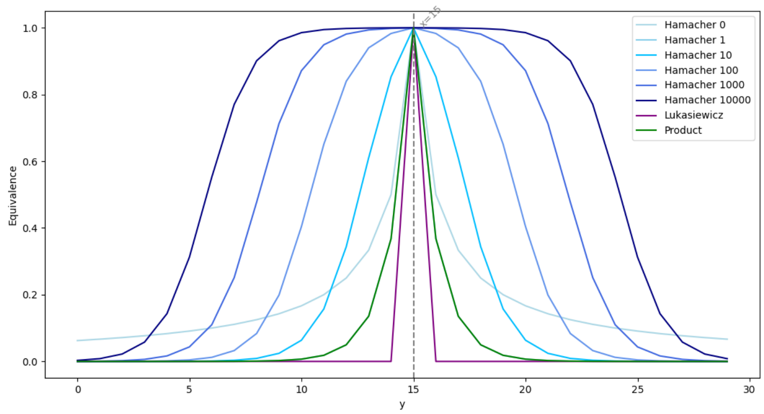

Fig. 1. T-equivalences $E(15, y)$ for product, Łukasiewicz's, and Hamacher's (with different λ) T-norms.

Now the question is how could we combine equivalences of one type effectively. The aggregation operator plays a crucial role by mathematically fusing diverse pieces of information into a consolidated representation. The choice of the aggregation function should be carefully tailored to the specific characteristics of the equivalences:

Definition 9. *(see e.g. [9]) A function*

$$\boldsymbol{A} : \cup_{n \in N} [0,1]^n \to [0,1]$$

is called an aggregation operator if it fulfills the following properties:

(A1) $\boldsymbol{A}(x_1, ..., x_n) \leq \boldsymbol{A}(y_1, ..., y_n)$ *whenever* $x_i \leq y_i$ *for all* $i \in 1, ..., n$*;*
(A2) $\boldsymbol{A}(x) = x$ *for all* $x \in [0,1]$*;*
(A3) $\boldsymbol{A}(0, ..., 0) = 0$ *and* $\boldsymbol{A}(1, ..., 1) = 1$*.*

Each aggregation operator $\mathbf{A}$ can be represented by a family $(\mathbf{A}_{(n)})_{n \in N}$ of n-ary operations, i.e. functions $\mathbf{A}_{(n)}(x_1, ..., x_n) = \mathbf{A}(x_1, ..., x_n)$.

Continuing, the definition of a subadditive function becomes paramount in metric-based fuzzy equivalence aggregation.

Definition 10. *(see e.g. [9]) A function* $F : [0,c]^n \to [0,c]$ *is subadditive on* $[0,c]$*, if the following inequality holds for all* $x_i, y_i \in [0,c]$ *with* $x_i + y_i \in [0,c]$*:*

$$F(x_1 + y_1, ..., x_n + y_n) \leq F(x_1, ..., x_n) + F(y_1, ..., y_n).$$

The next theorem formulates the main result for metric-based fuzzy equivalence aggregation, consolidating findings from [9] into a unified statement.

Theorem 2. *Consider a continious, Archimedian t-norm T with an additive generator g. Furthermore, let $\boldsymbol{A} : \cup_{n\in N}\,[0,1]^n \to [0,1]$ be an aggregation operator. Then $\boldsymbol{A}$ preserves T-transitivity of fuzzy relations $R_i : X_i \times X_i \to [0,1]$, where $i \in 1,...,n$, if and only if the aggregation operator $\boldsymbol{H} : \cup_{n\in N}\,[0,c]^n \to [0,c]$ defined by*

$$\boldsymbol{H}(z_1,...,z_n) = g(\boldsymbol{A}(g^{-1}(z_1),...,g^{-1}(z_n))) \qquad (1)$$

for all $n \in N$ and all $z_i \in [0,c]$ with $i \in 1,...,n$ is subadditive on $[0,c]^n$.

Proof. First let us prove that if $\mathbf{A}$ preserves T-transitivity then it leads to $\mathbf{H}$ being subadditive.

First, we show that if an aggregation operator $\mathbf{A}$ preserves T-transitivity then $T(\mathbf{A}(x_1,...,x_n),\mathbf{A}(y_1,...,y_n)) \le \mathbf{A}(T(x_1,y_1),...,T(x_n,y_n))$ holds for every $x_i, y_i \in [0,1]$, where $i \in \{1,\dots,n\}$.
Let $X_1 \times X_2 \times ... \times X_n$ contains at least three elements $a = (a_1,...,a_n)$; $b = (b_1,...,b_n)$; $c = (c_1,...,c_n)$. Then we define a T-transitive binary fuzzy relation R_i on X_i, $i \in \{1,\dots,n\}$, by $R_i(a_i,b_i) = R_i(b_i,a_i) = x_i$, $R_i(b_i,c_i) = R_i(c_i,b_i) = y_i$ and $T(x_i,y_i) = R_i(a_i,c_i) = R_i(c_i,a_i)$. Further let $R_i(d,d) = 1$ for all $d \in X_i$ and $R_i(d,e) = 0$ for all $d \ne e$ and for d or e different from a_i, b_i, c_i. For proving T-transitivity of R_i, we have to show that the following inequality holds for all $x,y,z \in X_i$: $T(R_i(x,y),R_i(y,z)) \le R_i(x,z)$. If any of the arguments x,y or z belongs to $X_i \setminus \{a_i,b_i,c_i\}$ the inequality is trivially fulfilled. Now we prove the T-transitivity of R_i for arguments $x,y,z \in \{a_i,b_i,c_i\}$:

$$T(R_i(a_i,b_i),R_i(b_i,c_i)) = T(x_i,y_i) = R_i(a_i,c_i),$$

$$T(R_i(b_i,c_i),R_i(c_i,a_i)) = T(y_i,T(x_i,y_i)) \le \min(x_i,y_i) \le x_i = R_i(b_i,a_i),$$

$$T(R_i(c_i,a_i),R_i(a_i,b_i)) = T(T(x_i,y_i),x_i) \le \min(x_i,y_i) \le y_i = R_i(c_i,b_i).$$

The other three inequalities are similar to the previous ones.
Thus, for arbitrary $x = (x_1,...,x_n)$, $y = (x_1,...,x_n) \in [0,1]^n$ we can find T-transitive binary fuzzy relations R_i on X_i which fulfills $x_i = R_i(a_i,b_i)$, $y_i = R_i(b_i,c_i)$. Therefore, taking into account that $\mathbf{A}$ preserves T-transitivity of fuzzy relations $R_i : X_i \times X_i \to [0,1]$, we conclude

$$\begin{gathered} T(\mathbf{A}(x_1,...,x_n),\mathbf{A}(y_1,...,y_n)) = \\ = T(\mathbf{A}(R_1(a_1,b_1),...,R_n(a_n,b_n)),\mathbf{A}(R_1(b_1,c_1),...,R_n(b_n,c_n))) = \\ = T(R(a,b),R(b,c)) \le R(a,c) = \mathbf{A}(R_1(a_1,c_1),...,R_n(a_n,c_n)) = \\ = \mathbf{A}(T(x_1,y_1),...,T(x_n,y_n)). \end{gathered}$$

Now we can rewrite

$$T(\mathbf{A}(x_1,...,x_n),\mathbf{A}(y_1,...,y_n)) \le \mathbf{A}(T(x_1,y_1),...,T(x_n,y_n))$$

as

$$\begin{aligned} & g^{(-1)}\left(\min\{g(0), g(\mathbf{A}(x_1,\dots,x_n)) + g(\mathbf{A}(y_1,\dots,y_n))\}\right) \le \\ & \le \mathbf{A}\left(g^{(-1)}\left(\min\{g(0), g(x_1)+g(y_1)\}\right),\dots,g^{(-1)}\left(\min\{g(0), g(x_n)+g(y_n)\}\right)\right) \end{aligned}$$

Consider some $n \in N$. Note that for arbitrary $u_i, v_i \in [0, c]$ and $i \in \{1, \ldots, n\}$ with $u_i + v_i \in [0, c]$ and $i \in \{1, \ldots, n\}$, there exist unique $x_i, y_i \in [0, 1]$ such that $u_i = g(x_i)$ and $v_i = g(y_i)$ for all $i \in \{1, \ldots, n\}$. Moreover, applying g to both sides of last inequality, we get

$$\min\{(g(0), g(\mathbf{A}(x_1, ..., x_n)) + g(\mathbf{A}(y_1, ..., y_n)))\} \geq$$

$$\geq g(\mathbf{A}(g^{(-1)}(u_1 + v_1), ..., g^{(-1)}(u_n + v_n)))$$

Now let's define $\mathbf{H}_{(n)} : [0, c]^n \to [0, c]$ by

$$\mathbf{H}_{(n)}(u_1, ..., u_n) = g(\mathbf{A}(g^{(-1)}(u_1), ..., g^{(-1)}(u_n))), \tag{2}$$

then $\mathbf{H}_{(n)}$ is non-decreasing mapping fulfilling

$$\mathbf{H}_{(n)}(0, ..., 0) = g(\mathbf{A}(1, ..., 1)) = g(1) = 0,$$

$$\mathbf{H}_{(n)}(c, ..., c) = g(\mathbf{A}(0, ..., 0)) = g(0) = c,$$

$$\mathbf{H}_{(n)}(u_1 + v_1, ..., u_n + v_n) \leq \min\{(g(0), \mathbf{H}_{(n)}(u_1, ..., u_n) + \mathbf{H}_{(n)}(v_1, ..., v_n))\} \leq$$

$$\leq \mathbf{H}_{(n)}(u_1, ..., u_n) + \mathbf{H}_{(n)}(v_1, ..., v_n),$$

i.e., for arbitrary $\mathbf{H}_{(n)}$ is n-ary aggregation operator, which is subadditive on $[0, c]$.

Now let us prove that if $\mathbf{H}$ is subadditive then $\mathbf{A}$ preserves T-transitivity of fuzzy relations.
For a given subadditive aggregation operator $\mathbf{H} : \bigcup_{n \in N} [0, c]^n \to [0, c]$, define $\mathbf{A} : \bigcup_{n \in N} [0, 1]^n \to [0, 1]$ by

$$\mathbf{A}(x_1, ..., x_n) = g^{(-1)}(\mathbf{H}(g(x_1), ..., g(x_n))).$$

Evidently, $\mathbf{A}$ is an aggregation operator. Due to subadditivity of $\mathbf{H}$ we get

$$\mathbf{H}(u_1 + v_1, ..., u_n + v_n) \leq \mathbf{H}(u_1, ..., u_n) + \mathbf{H}(v_1, ..., v_n)$$

and that inequality can be rewritten using $\mathbf{A}$ as

$$g(\mathbf{A}(g^{(-1)}(u_1 + v_1), ..., g^{(-1)}(u_n + v_n))) \leq$$

$$\leq g(\mathbf{A}(g^{(-1)}(u_1), ..., g^{(-1)}(u_n))) + g(\mathbf{A}(g^{(-1)}(v_1), ..., g^{(-1)}(v_n)))$$

by applying $g^{(-1)}$ for both sides of the inequality we get

$$\mathbf{A}(T(x_1, y_1), ..., T(x_n, y_n)) \geq T(\mathbf{A}(x_1, ..., x_n), \mathbf{A}(y_1, ..., y_n)).$$

Now we will show, that because of the last inequality, $\mathbf{A}$ preserves T-transitivity. Therefor we have to show that $\tilde{R} = \mathbf{A}(R_1, ..., R_n)$ is T-transitive for some binary,

T-transitive relations R_i on X_i with $i \in \{1, \dots, n\}$ and some $n \in N$.
Consider arbitrary $a, b, c \in X_1 \times ... \times X_n$, then we get

$$T(\tilde{R}(a,b), \tilde{R}(b,c)) =$$
$$= T(\mathbf{A}(R_1(a_1,b_1), ..., R_n(a_n,b_n)), \mathbf{A}(R_1(b_1,c_1), ..., R_n(b_n,c_n))) \leq$$
$$\leq \mathbf{A}(T(R_1(a_1,b_1), R_1(b_1,c_1)), ..., T(R_n(a_n,b_n), R_n(b_n,c_n))) \leq$$
$$\leq \mathbf{A}(R_1(a_1,c_1), ..., R_n(a_n,c_n)) = \tilde{R}(a,c).$$

Thus we obtain the following formula for finding an aggregation operator $\mathbf{A}$ for any fuzzy equivalence relation.

Corollary 1. *Let $\boldsymbol{A} : \cup_{n \in N} [0,1]^n \to [0,1]$ be an aggregation operator defined as:*

$$\boldsymbol{A}(x_1, ..., x_n) = g^{(-1)}(\min(g(0), \sum_{i=1}^{n} p_i g(x_i)),$$

where

- *g - additive generator;*
- *p_i - weights such that $1 \leq \sum_{i=1}^{n} p_i$,*

then $E((x_1, ..., x_n), (y_1, ..., y_n)) = \boldsymbol{A}(E_1(x_1,y_1), E_2(x_2,y_2), ..., E_n(x_n,y_n))$ is the fuzzy equivalence relation (T-equivalence) if E_i are fuzzy equivalence relations (T-equivalence) for all $i = 1, ..., n$.

Some examples of aggregation operators $\mathbf{A}$ for previously mentioned fuzzy equivalences, where p_i are weights such that $1 \leq \sum_{i=1}^{n} p_i$:

- $\mathbf{A}_p(E_p(x_1,y_1), E_p(x_2,y_2), ..., E_p(x_n,y_n)) = e^{-\sum_{i=1}^{n} p_i d(x_i,y_i)}$ aggregation operator of E_p defined as in Example 1;
- Aggregation operator of E_{L} defined as in Example 2 is defined as follows:

$$\mathbf{A}_{\text{L}}(E_{\text{L}}(x_1,y_1), ..., E_{\text{L}}(x_n,y_n)) = 1 - \min(1, \sum_{i=1}^{n} p_i \cdot (1 - \max(0, 1 - d(x_i,y_i))));$$

- Aggregation operator of E_H defined as in Example 3 is defined as follows

$$\mathbf{A}_{H_\lambda}(E_H(x_1,y_1), E_H(x_2,y_2), ..., E_H(x_n,y_n)) = \begin{cases} \frac{1}{1+\sum_{i=1}^{n} p_i d(x_i,y_i)}, & \lambda = 0 \\ \frac{\lambda}{e^{\sum_{i=1}^{n} p_i d(x_i,y_i)} - 1 + \lambda}, & \lambda \neq 0. \end{cases}$$

4 Fuzzy inequality relations

We proceed with an overview of basic definitions and results on fuzzy inequality relations, dual operation to equivalence relations.

Definition 11. *A fuzzy binary relation D on a set S is called a fuzzy inequality relation with respect to a t-conorm S (or S-inequality), if and only if the following three axioms are fulfilled for all $x, y, z \in M$:*

1. $D(x,x)=0$ *reflexivity;*
2. $D(x,y)=D(y,x)$ *symmetry;*
3. $D(x,z)\leq S(D(x,y),D(y,z))$ *S-transitivity.*

The following result establishes principles of construction of fuzzy inequality relations using pseudo-metrics.

Theorem 3. *Let S be a continuous Archimedean t-conorm with an additive generator g. For any pseudo-metric d, the mapping*

$$D_d(x,y)=g^{(-1)}(\min(d(x,y),g(1)))$$

is a S-inequality.

Let us consider 3 examples of fuzzy inequalities that are constructed using additive generators of previously mentioned t-conorms:

Example 4. [S-inequality for product t-conorm]

$$D_P(x,y)=1-e^{-d(x,y)}.$$

Example 5. [S-inequality for Łukasiewicz's t-conorm]

$$D_L(x,y)=\min(1,d(x,y)).$$

Example 6. [S-inequality for Hamacher's t-conorm]

$$D_{H_\lambda}(x,y)=\begin{cases}\frac{d(x,y))}{1+d(x,y)},\ if\ \lambda=0\\ 1-\frac{\lambda}{e^{d(x,y)}+\lambda-1},\ \lambda\in]0;\infty[.\end{cases}$$

In the figure 2 we see the graphics of these inequality relations.

Similar to fuzzy equivalences, we can construct an aggregation operator for fuzzy inequalities as well. Unlike aggregation operator of fuzzy equivalence, the aggregation operator of fuzzy inequality should remain property of S-transitivity. The proof is similar to one mentioned above, and we get the following formula for aggregation operator of fuzzy inequalities:

Theorem 4. *Consider a continuous, Archimedian t-conorm S with an additive generator g. Furthermore, let $\boldsymbol{A}:\cup_{n\in N}[0,1]^n\rightarrow[0,1]$ be an aggregation operator. Then $\boldsymbol{A}$ preserves S-transitivity of fuzzy relations $R_i:X_i\times X_i\rightarrow[0,1]$, where $i\in 1,...,n$, if and only if the aggregation operator $\boldsymbol{H}:\cup_{n\in N}[0,c]^n\rightarrow[0,c]$ defined by*

$$\boldsymbol{H}(z_1,...,z_n)=g(\boldsymbol{A}(g^{-1}(z_1),...,g^{-1}(z_n)))\tag{3}$$

for all $n\in N$ and all $z_i\in[0,c]$ with $i\in 1,...,n$ is subadditive on $[0,c]^n$.

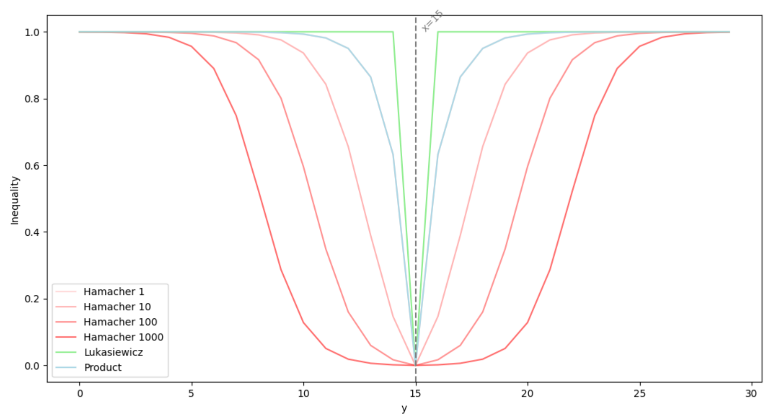

Fig. 2. S-inequality $D(15, y)$ for product, Łukasiewicz's, and Hamacher's (with different λ) T-conorms.

Corollary 2. *Let* $\boldsymbol{A} : \cup_{n \in N} [0,1]^n \to [0,1]$ *be an aggregation operator defined as:*

$$\boldsymbol{A}(x) = g^{(-1)}(\min(g(1), \sum_{i=1}^{n} p_i g(x_i))),$$

where

- g *- additive generator of a t-conorm* S*;*
- p_i *- weights such that* $1 \le \sum_{i=1}^{n} p_i$*,*

then $D((x_1, ..., x_n), (y_1, ..., y_n)) = \boldsymbol{A}(D_1(x_1, y_1), D_2(x_2, y_2), ..., D_n(x_n, ..., x_n))$ *is the fuzzy inequality relation (*S*-inequality) if* D_i *are fuzzy inequality relations (*S*-inequalities) for all* $i = 1, ..., n$*.*

Some examples of aggregation operators A for previously mentioned fuzzy inequalities:

- $A_p(D_p(x_1, y_1), D_p(x_2, y_2), ..., D_p(x_n, y_n)) = 1 - e^{-\sum_{i=1}^{n} p_i d(x_i, y_i)}$ aggregation operator of D_p defined as in Example 4;
- $A_Ł(D_Ł(x_1, y_1), D_Ł(x_2, y_2), ..., D_Ł(x_n, y_n)) = \min(1, \sum_{i=1}^{n} p_i d(x_i, y_i))$ aggregation operator of $D_Ł$ defined as in Example 5;
- Aggregation operator of D_H defined as in Example 6 is defined as follows:

$$A_H(D_H(x_1, y_1), ..., D_H(x_n, y_n)) = \begin{cases} \frac{\sum_{i=1}^{n} p_i d(x_i, y_i))}{1 + \sum_{i=1}^{n} p_i d(x_i, y_i))}, & \lambda = 0 \\ 1 - \frac{\lambda}{e^{\sum_{i=1}^{n} p_i d(x_i, y_i)} - 1 + \lambda}, & \lambda \neq 0. \end{cases}$$

5 Conclusions

In summary, our research has provided valuable insights into the theoretical aspects of fuzzy logic, particularly focusing on fuzzy equivalence and inequality relations, as well as their aggregation operators. We have contributed by proving formulas for constructing aggregation operators for fuzzy equivalences and deriving similar concepts for aggregation operators for inequality relations. Through rigorous analysis, we have deepened our understanding of metric-based fuzzy relations, laying a solid foundation for future research and practical applications in decision-making processes. We believe that our contributions will inspire further advancements in this field and pave the way for innovative applications of fuzzy relations in various domains.

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