Fuzzy integrals and if then rules: the bipolar case

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Abstract. This paper summarizes the contribution of the late Michio Sugeno to non-additive integrals. Moreover, based on the existence of a natural interpretation of Sugeno integral as a set of fuzzy rules involving thresholds, it proposes preliminary results on the possibility of interpreting the bipolar fuzzy integral in terms of if-then rules.

Keywords: Sugeno integral · if-then rules · bipolarity

1 Introduction

Michio Sugeno has made landmark contributions in at least two very different areas of fuzzy logic: first, at the theoretical level with the invention of fuzzy integral [?] and, the second one, more applied, with the design of a particular type of fuzzy rule-based non-linear systems and controllers, and the study of their stability [?][?][?]. He also published noticeable works on a variety of topics that for instance range from showing a relation between a Choquet integral equation and the Abel integral equation (and thus fractional calculus [?]), to studying the brain activity when understanding natural language [?] as well as the generation of meaningful linguistic descriptions of phenomena [?]. In the following, we only focus on Sugeno integrals.

The first part of this article is a reminder of the fuzzy integral introduced by Michio Sugeno during his thesis [?]. The second part concerns the representation of qualitative data. We use the results concerning the relation between if-then rules and Sugeno integrals to study the possibility of a rule-based representation of a bipolar version of fuzzy integral, especially the Cumulative Prospect Theory Sugeno (CTPS) integral [?], the extension of the Sugeno integral to bipolar scales.

The paper is organised as follows. The next section is devoted to highlight the early contributions of Michio Sugeno as they appear in his Ph.D. thesis. The third section recalls the use of Sugeno integrals for representing qualitative data in terms of selection and elimination rules. The last section deals with a bipolar variant of Sugeno integral: the CPTS integral. It proposes first results on the interpretation of this bipolar qualitative integral in terms of if-then rules.

2 The Sugeno integral

Sugeno's Ph.D. thesis is a major contribution to non-additive integrals developed in parallel to classical measure theory. The intuitions guiding the latter are then carried over to a qualitative setting.

2.1 The dissertation submitted in 1974 by Michio Sugeno

The dissertation submitted in partial fulfillment of the requirements for the degree of doctor of engineering by Michio Sugeno [?] (see [?] for a partial summary account) is highly remarkable in many respects (in spite of its relative brevity: 124 + viii pages):

- **A breakthrough in fuzzy set research of the time**:
  In the mid-1970s, fuzzy set research was dominated by the idea of membership functions and their introduction and use in whatever domain of mathematics where the notion of set made sense. At the time, even the notion of fuzzy number (as a fuzzy set of numerical values) was in its infancy. Nobody at that time was thinking of a *grade of fuzziness* (in the sense used by Sugeno [?]) for evaluating a statement such as "x belongs to A", where x is unknown and A a known set. The increasing set function instrumental in this task appeared to be more consistent with human evaluation of situations than the stronger additivity hypothesis of probability theory.
- **The idea of** max**-decomposable fuzzy measure**:
  The second example of a fuzzy measure given by Sugeno in his thesis [?] (page 12) is the so-called *F-additive fuzzy measure* which is nothing but the possibility measure proposed later on by Zadeh [?]. However, Sugeno did not develop F-additive measures for themselves. He rather introduced λ-fuzzy measures g_λ that remain closer to probabilities and satisfy the relaxed additivity property:

  $$\forall A, B \text{ such that } A \cap B = \emptyset, g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$$

  with $\lambda > -1$. Such fuzzy measures turn out to be special cases of Shafer belief functions [?] for $\lambda \geq 0$ and plausibility functions for $-1 < \lambda \leq 0$. They obviously reduce to probabilities when $\lambda = 0$. These λ-fuzzy measures are never possibility nor necessity measures [?].
- **Sugeno's fuzzy integral**:
  the author provided a qualitative max-min counterpart to classical quantitative measure theory by offering a fuzzy measure theory. This integral is presented with details in the next section.
- **Application in pattern regognition**:
  Sugeno's thesis [?] contains a chapter dedicated to applications of fuzzy integrals (Chapter 5). Even if Sugeno integrals have gained recognition as a valuable tool in multiple criteria analysis for a long time [?], the applications considered in the thesis are more in the realm of subjective evaluation of fuzzy objects [?,?] and machine learning. Fuzzy integrals are used for grading similarity of patterns.

– **An artificial intelligence perspective**:
Fuzzy logic has developed outside mainstream artificial intelligence, although a latent influence of the latter on fuzzy set research was at work. For instance, Mamdani's fuzzy rule-based controllers can be regarded as fuzzy expert systems as they were considered from the start as such [**?**]. Even if it may sound more surprizing, Sugeno also predicted that fuzzy integrals should play a role in the future of artificial intelligence. Indeed the last sentence of the thesis is "It is particularly hoped that this research will serve in future for the studies of artificial intelligence." This sentence could turn out to be considered as prophetic if qualitative methods could gain a larger place in artificial intelligence in the future.

2.2 The various representations of Sugeno integral

We consider a finite set of n criteria, $\mathcal{C} = \{1, \cdots, n\}$. We consider L a rating scale which is a totally ordered structure (L can also be an interval of $\mathbb{R}$). For our purposes, we assume that L has a top element denoted by 1, and a bottom one denoted by 0. The rating scale L is assumed to be equipped with an involutive negation, denoted by $\nu(\cdot)$, which reverses the order of the scale; in particular $\nu(0) = 1, \nu(1) = 0$.

The objects or alternatives are vectors $x = (x_1, \ldots, x_n) \in L^n$ where x_i is the local evaluation (partial evaluation of x with respect to criterion i).

Sugeno integrals are defined on the basis of increasing set functions Sugeno called fuzzy measures and that are also known as Choquet capacities. These set functions contain the information of the model.

Namely, a capacity (or fuzzy measure) is a set function $\mu : 2^{\mathcal{C}} \to L$ such that $\mu(\emptyset) = 0$, $\mu(\mathcal{C}) = 1$ and $\forall A, B \subseteq \mathcal{C}$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.

The first definition of Sugeno integral given in [**?**] is:

$$S_\mu(x) = \max_{\alpha \in L} \min(\alpha, \mu(x \geq \alpha)), \text{ where } \mu(x \geq \alpha) = \mu(\{i \in \mathcal{C} | x_i \geq \alpha\}).$$

in the discrete case the fuzzy integral was a median [**?**]. Namely, and assume the x_i's are ranked in ascending order $x_1 \leq \cdots \leq x_n$, then $S_\mu(x)$ is the median of the $2n - 1$ numbers

$$\{x_1, \cdots, x_n\} \cup \{\mu(\{n\}), \cdots, \mu(\{2, \cdots, n\})\}$$

Thus a fuzzy integral may be seen as a "weighted median" of $\{x_1, \cdots, x_n\}$ (while Lebesgue integral is a weighted average). In particular it is clear that

$$\min_{x_i : i \in \mathcal{C}} x_i \leq S_\mu(x) \leq \max_{x_i : i \in \mathcal{C}} x_i.$$

Sugeno integral has several other equivalent formulations:

$$S_\mu(x) = \max_{A \subseteq \mathcal{C}} \min(\mu(A), \min_{i \in A} x_i)$$

and, letting $\bar{A}$ be the complement of set A:

$$S_\mu(x) = \min_{A \subseteq \mathcal{C}} \max(\mu(\bar{A}), \max_{i \in A} x_i).$$

The qualitative Möbius transform of the capacity μ is the set function containing the minimum information to reconstruct the capacity μ. Formally, the qualitative Möbius transform of the capacity μ is defined by

$$\mu_\#(T) = \begin{cases} \mu(T) & \text{if } \mu(T) > \max_{i \in T} \ \mu(T \setminus \{i\}) \\ 0 & \text{else} \end{cases}$$

We have $\forall A \subseteq \mathcal{C}, \quad \mu(A) = \max_{T \subseteq A} \mu_\#(T)$. The sets T for which $\mu_\#(T) > 0$ are called the focal sets of μ. The set of focal elements of μ is denoted by $\mathcal{F}(\mu)$. The Möbius transform of the capacity is enough to define the Sugeno integral, namely we can replace the power set of $\mathcal{C}$ by $\mathcal{F}(\mu)$ in its last but one above expression, and by $\mathcal{F}(\mu^c)$ in its last above expression, where μ^c is the conjugate of μ: $\mu^c(A) = \nu(\mu(\bar{A}))$.

Sugeno integrals have early attracted the interest of many researchers, both theoretical [**?**,**?**,**?**] and application oriented, as witnessed by the early edited volume on Sugeno integrals [**?**].

Sugeno integral can serve as a substitute to expected utility and weighted average, more generally to Choquet integral, in decision evaluation methods [**?**,**?**,**?**,**?**]. The specificity of Sugeno integral with respect to the other utility funtionals is that it can be applied to qualitative ratings and degrees of uncertainty, not requiring the use of precise numbers. Indeed Sugeno integral only involves maximum and minimum operations, and only requires a (possibly finite) totally ordered scale of values. In order to lay bare the significance of Sugeno integral as a tool for decision evaluation, axiomatic justifications of its use to rank-order acts in decision under uncertainty, or alternatives in multi-criteria decision have been provided in [**?**,**?**,**?**,**?**]. See [**?**] for a survey of existing results on Sugeno integrals as rational decision evaluation methods under uncertainty.

3 Qualitative data analysis and the if-then rule interpretation of Sugeno integral

Given a set of data made of tuples gathering partial evaluations according to different evaluation criteria together with the corresponding global evaluation that respect the previous condition, we face the problem of determining Sugeno integrals agreeing with such data [**?**,**?**]. Several distinct families of Sugeno integrals may be necessary for covering a set of data in case several aggregative behaviors are at work in the data [**?**].

Once we have learned a Sugeno integral from data it is possible to exhibit an equivalent representation in terms of if-then rules, which lays bare how the global evaluation depends on the evaluation of each criterion. Indeed, Greco *et al.* [**?**] have shown that Sugeno integrals can be represented by selection or elimination rules involving the same threshold for all the conditions and the conclusion. Thus, Sugeno integral appears to be equivalent to a set of rules where each rule involves a single threshold [**?**].

Greco *et al.* [**?**] had already observed that Sugeno integrals have limited expressive power: they can only represent selection and elimination rules involving the same threshold for all attributes. In [**?**], Sugeno integral appears to be equivalent to a single-valued set of rules. We are interested in a construction of the set of rules equivalent to a Sugeno integral.

The inequality $S_\mu(x) \geq \alpha$ is equivalent to the existence of a focal set $T \in \mathcal{F}(\mu)$ such that $\forall i \in T, x_i \geq \alpha$, so we obtain the following selection rule.

Proposition 1. *Any focal set T of a capacity μ with weight $\mu_\#(T)$ corresponds to the selection rule :*

$$R^s_T : \textit{If } x_i \geq \mu_\#(T) \textit{ for all } i \in T \textit{ then } S_\mu(x) \geq \mu_\#(T).$$

Conversely, if we consider a single threshold selection rule, we can construct a focal set of the associated capacity.

Symmetrically, the inequality $S_\mu(x) \leq \alpha$ is equivalent to the existence of a focal set $F \in \mathcal{F}(\mu^c)$ with $\mu^c(F) \geq \nu(\alpha)$ such that $\forall i \in F, x_i \leq \alpha$. So we obtain the following elimination rule.

Proposition 2. *Any focal set F of the the conjugate capacity of μ, μ^c, corresponds to the elimination rule:*

$$R^e_F : \textit{If } x_i \leq \nu(\mu^c_\#(F)) \textit{ for all } i \in F \textit{ then } S_\mu(x) \leq \nu(\mu^c_\#(F)).$$

Conversely, if we start from a single threshold elimination rule, we can construct the focal set of the associated conjugate capacity.

A Sugeno integral involving a capacity μ is thus equivalent to a set of selection rules, one per focal set of μ, or equivalently to a set of elimination rules, one per focal set of μ^c.

4 The Cumulative Prospect Theory Sugeno integral

The idea of representing Sugeno integrals by thresholded rules can be extended to asymmetric bipolar Sugeno integrals, such as CPTS integrals [**?**] (short for Cumulative Prospect Theory Sugeno integrals), mimicking the cardinal preference model called Prospect Theory [**?**]. In case decisions are evaluated according to positive and negative aspects, using an evaluation scale that is only used to compare alternatives is insufficient. The evaluation scale should be bipolar. A bipolar scale consists of a positive part, a negative part and a neutral level which is a point considered neither positive nor negative.

In the CPTS model, first the positive and negative parts of the evaluation are treated separately. Then, they are combined using the so-called symmetric maximum introduced in [?] .

In [?] the last author and Sugeno have presented a method for the elicitation of CPTS-integrals agreeing with a data set composed of pairs concerning some criteria and a global evaluation. Moreover the set of fuzzy measures that are solutions of this inverse problem is identified. When the elicitation of one CPTS-integral is not possible, a set of family of CPTS-integrals is proposed. The aim of this paper is to address the next step, namely we investigate the problem of interpreting a CPTS integral as a set of if-then rules.

4.1 Bipolar scales

In [?], Grabisch proposed a symmetric extension of Sugeno integral to signed values. This extension is purely ordinal and in the spirit of the symmetric (Šipos) extension of Choquet integral [?]. It is based on extending the maximum and minimum operations so that the algebraic structure is close to a ring. Let us briefly recall some definitions.

A linearly ordered set $\{L^+, \leq\}$, with bottom and top elements denoted 0 and 1, is considered as a scale of positive levels. The negative levels are modeled as a reversed mirror image of L^+, namely, $L^- = \{-\alpha \mid \alpha \in L^+\}$. Moreover the function $-(\cdot)$ is involutive, i.e., $-(-\alpha) = \alpha$. The bottom and top of L^- are -1 and -0, respectively, where -0 is equal to 0. L denotes the union of L^+ and $L^-$and stands as a bipolar scale. Thus the top of L is 1 and its bottom is -1.

The absolute value of $\alpha \in L$ is denoted by $|\alpha|$ where

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \in L^+ \\ -\alpha & \text{if } \alpha \in L^- \end{cases}$$

When $\alpha, \beta \in L$, the symmetric extension of the max operation is denoted by $\circledvee$ and defined as follows [?]:

$$\alpha \circledvee \beta = \begin{cases} -\max(|\alpha|, |\beta|) \text{ if } \beta \neq -\alpha \text{ and } \max(|\alpha|, |\beta|) = \text{ either } -\alpha \text{ or } -\beta; \\ 0 \text{ if } \beta = -\alpha; \\ \max(|\alpha|, |\beta|) \text{ otherwise.} \end{cases}$$

Thus, except if $\beta = -\alpha$, $\alpha \circledvee \beta$ is equal to the level (positive or negative) with maximal absolute value among α and β, namely

- If $\alpha, \beta \in L^+$ then $\alpha \circledvee \beta = \max(\alpha, \beta)$.
- If $\alpha, \beta \in L^-$ then $\alpha \circledvee \beta = -\max(-\alpha, -\beta)$.
- If $\alpha \in L^+, \beta \in L^-$, $\alpha \neq -\beta$ then $\alpha \circledvee \beta = \alpha$ if $\alpha > -\beta$ and β if $\alpha < -\beta$.
- If $\alpha \in L^-, \beta \in L^+$, $\alpha \neq -\beta$ then $\alpha \circledvee \beta = \alpha$ if $-\alpha > \beta$ and β if $-\alpha < \beta$.
- If $\alpha = -\beta$ then $\alpha \circledvee \beta = 0$.

Based on this symmetric bipolar version of the max operation, several symmetric extensions of the Sugeno integral have been proposed in [?]. We consider here a qualitative version of the Šipos integral.

We consider a function $x : \mathcal{C} \to L$ and two capacities, $\mu^+ : 2^{\mathcal{C}} \to L^+$ and $\mu^- : 2^{\mathcal{C}} \to L^+$ (one for the positive part and one for the negative part). We define $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ where the maximum is calculated component by component.

The Cumulative Prospect Theory S-integral [**?**,**?**] of x with respect to μ^- and μ^+ is defined as

$$CPTS_{\mu^-,\mu^+}(x) = S_{\mu^+}(x^+) \circledvee -S_{\mu^-}(x^-).$$

It could be called asymmetric bipolar Sugeno integral. In case $\mu^+ = \mu^-$, it corresponds to the symmetric Sugeno integral [**?**].

Property 1.

- $CPTS_{\mu^-,\mu^+}(x) = \alpha > 0$ iff $S_{\mu^-}(x^-) < S_{\mu^+}(x^+) = \alpha$.
- $CPTS_{\mu^-,\mu^+}(x) = \alpha < 0$ iff $S_{\mu^+}(x^+) < S_{\mu^-}(x^-) = -\alpha$.
- $CPTS_{\mu^-,\mu^+}(x) = 0$ iff $S_{\mu^+}(x^+) = S_{\mu^-}(x^-)$.

Proof. We have $S_{\mu^+}(x^+) \in L^+$ and $-S_{\mu^-}(x^-) \in L^-$ so we have

- if $S_{\mu^+}(x^+) \neq S_{\mu^-}(x^-)$:
  $S_{\mu^+}(x^+) \circledvee -S_{\mu^-}(x^-) = \begin{cases} S_{\mu^+}(x^+) \text{ if } S_{\mu^+}(x^+) > S_{\mu^-}(x^-) \\ -S_{\mu^-}(x^-) \text{ if } S_{\mu^+}(x^+) < S_{\mu^-}(x^-). \end{cases}$
- if $S_{\mu^+}(x^+) = S_{\mu^-}(x^-)$: $S_{\mu^+}(x^+) \circledvee -S_{\mu^-}(x^-) = 0$

The other equivalences are proved similarly.

Note that this property was used to elicit the CPTS integral [**?**]. Let us briefly recall this method. Given a vector x with its evaluation $\alpha \in L$, we want to elicit capacities μ^+ and μ^- such that $CPTS_{\mu^+,\mu^-}(x) = \alpha \in L$.

If $\alpha \geq 0$, we need to identify μ^+ such that $S_{\mu^+}(x^+) = \alpha$ and μ^- such that $S_{\mu^-}(x^-) = \beta$ with $0 \leq \beta < \alpha$. We must try all such possible levels β.

If $\alpha \leq 0$ we need to identify μ^+ and μ^- such that $S_{\mu^-}(x^-) = -\alpha$ and $S_{\mu^+}(x^+) = \beta$ with $0 \leq \beta < -\alpha$. We must try all such possible levels β.

In [**?**], there is also a property which characterises the cases for which there exists a CPTS integral that represents the dataset. When the CPTS integral exists, we can calculate the bounds of the capacities μ^+ and μ^- solutions of the elicitation problem, applying previous results concerning Sugeno integral [**?**].

4.2 An if-then rule interpretation

According to the property presented above, we have the following equivalences:

- $CPTS_{\mu^-,\mu^+}(x) \geq \alpha > 0$ iff $S_{\mu^+}(x^+) \geq \alpha$ and $S_{\mu^+}(x^+) > S_{\mu^-}(x^-)$.
- $0 \leq CPTS_{\mu^-,\mu^+}(x) \leq \alpha$ iff $S_{\mu^+}(x^+) \leq \alpha$ and $S_{\mu^+}(x^+) > S_{\mu^-}(x^-)$.
- $-\alpha > -CPTS_{\mu^-,\mu^+}(x) > 0$ iff $S_{\mu^-}(x^-) \leq -\alpha$ and $S_{\mu^+}(x^+) < S_{\mu^-}(x^-)$.
- $-CPTS_{\mu^-,\mu^+}(x) \geq -\alpha > 0$ iff $S_{\mu^-}(x^-) \geq -\alpha$ and $S_{\mu^+}(x^+) < S_{\mu^-}(x^-)$.

Accordingly, we can induce if-then rules from a CPTS integral:

Proposition 3. *Any pair of focal sets (T, F) where*

- T *is a focal set of* μ^+,
- F *is a focal set of* $(\mu^-)^c$,
- $\nu((\mu^-)^c_\#(F)) < \mu^+_\#(T)$

induces the rule
if $x_i^+ \geq \mu^+_\#(T), \forall i \in T$ *and* $x_i^- \leq \nu((\mu^-)^c_\#(F)), \forall i \in F$ *then* $CPTS(x) \geq \mu^+_\#(T)$.

Proof. Any focal set T of μ^+ induces the rule if $x_i^+ \geq \mu^+_\#(T)$ for all $i \in T$ then $S_{\mu^+}(x^+) \geq \mu^+_\#(T)$.

Any focal set F of $(\mu^-)^c$ induces the rule if $x_i^- \leq \nu((\mu^-)^c_\#(F))$ for all $i \in F$ then $S_{\mu^-}(x^-) \leq \nu((\mu^-)^c_\#(F))$.

So we have $S_{\mu^-}(x^-) \leq \nu((\mu^-)^c_\#(F)) < \mu^+_\#(T) \leq S_{\mu^+}(x^+)$ and it implies $CPTS(x) = S_{\mu^+}(x^+) \geq \mu^+_\#(T)$.

Proposition 4. *Any pair of focal sets (T, F) where*

- T *is a focal set of* μ^{-c} *with* $\mu^{-c}_\#(T) = 1$
- F *is a focal set of* μ^{+c}

induces the rule:
If $x_i^- = 0$ *for all* $i \in T$ *and* $x_i^+ \leq \nu((\mu^+)^c_\#(F))$ *for all* $i \in F$ *then* $0 \leq CPTS_{\mu^-,\mu^+}(x) \leq \nu((\mu^+)^c_\#(F))$.

Proof. If $x_i^- = 0$ for all $i \in T$ then $S_{\mu^-}(x^-) = \min_{A \subseteq \mathcal{C}} \max(\mu(\bar{A}), \max_{i \in A} x_i^-) \leq \max(\mu^-(\bar{T}), \max_{i \in T} x_i^-) = 0$. So $CPTS(x) = S_{\mu+}(x^+) \circledvee 0 = S_{\mu+}(x^+)$.
The hypothesis $x_i^+ \leq \nu((\mu^+)^c_\#(F))$ for all $i \in F$ implies $S_{\mu+}(x^+) \leq \nu((\mu^+)^c_\#(F))$.

Proposition 5. *Any pair of focal sets (T, F) where*

- T' *is a focal set of* μ^{+c} *with* $\mu^{+c}_\#(T') = 1$,
- F *is a focal set of* μ^{-c}

induces the rule
If $x_i^+ = 0$ *for all* $i \in T'$ *and* $x_i^- \leq \nu((\mu^-)^c_\#(F))$ *for all* $i \in F$
then $0 \leq -CPTS_{\mu^-,\mu^+}(x) \leq \nu((\mu^-)^c_\#(F))$.

Proof. If $x_i^+ = 0$ for all $i \in T'$ then $S_{\mu^+}(x^+) = \min_{A \subseteq \mathcal{C}} \max(\mu(\bar{A}), \max_{i \in A} x_i^+) \leq \max(\mu^+(\bar{T'}), \max_{i \in T'} x_i^+) = 0$. So $CPTS(x) = 0 \circledvee -S_{\mu^-}(x^-) = -S_{\mu-}(x^-)$.
The hypothesis $x_i^- \leq \nu((\mu^-)^c_\#(F))$ for all $i \in F$ implies $S_{\mu^-}(x^-) \leq \nu((\mu^-)^c_\#(F))$.
So $0 \leq -CPTS(x) = S_{\mu-}(x^-) \leq (1 - (\mu^-)^c_\#(F))$.

Proposition 6. *Any pair of focal sets (T, F) where*

- T *is a focal set of* μ^-,
- F *is a focal set of* $(\mu^+)^c$,
- $\nu((\mu^+)^c_\#(F)) < \mu^-_\#(T)$

induces the rule
if $x_i^- \geq \mu^-_{\#}(T), \forall i \in T$ *and* $x_i^+ \leq \nu((\mu^+)^c_{\#}(F)), \forall i \in F$ *then* $CPTS(x) \leq -\mu^+_{\#}(T)$.

Proof. If $x_i^- \geq \mu^-_{\#}(T)$ for all $i \in T$ then $S_{\mu^-}(x^-) \geq \mu^-_{\#}(T)$, i.e.,
$-S_{\mu^-}(x^-) \leq -\mu^-_{\#}(T)$.
If $x_i^+ \leq \nu((\mu^+)^c_{\#}(F))$ for all $i \in F$ then $S_{\mu^+}(x^+) \leq \nu((\mu^+)^c_{\#}(F))$. We have $\nu((\mu^+)^c_{\#}(F)) < \mu^-_{\#}(T)$ so $S_{\mu^+}(x^+) < S_{\mu^-}(x^-)$.
We have $CPTS(x) = -S_{\mu^-}(x^-) \leq -\mu^-_{\#}(T)$.

Example 1. We consider 4 criteria, the scale $L = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and the following capacities :

focal set	$\{1,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
μ^+	4	3	5

focal set	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,4\}$
μ^{+c}	2	1	5	1	5	5
$5-\mu^{+c}$	3	4	0	4	0	0

foc	$\{1,3\}$	$\{2,3\}$	$\{2,4\}$	$\{1,2,3,4\}$
μ^-	3	2	1	5

focal set	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{2,3\}$	$\{3,4\}$
μ^{-c}	3	2	4	2	5	5	5
$5-\mu^{-c}$	2	3	1	3	0	0	0

- We consider the focal set $T = \{1,3\}$ of μ^+, with $\mu^+_{\#}(T) = 4$.
- We consider the focal set $F = \{2\}$ of $\mu^{-,c}$, with $\nu((\mu^-)^c_{\#}(F)) = 3$.
- We have $\nu((\mu^-)^c_{\#}(F)) < \mu^+_{\#}(T)$

The CPTS integral generates the rule
if $x_1^+ \geq 4$ and $x_3^+ \geq 4$ and $x_2^- \leq 3$ then $CPTS(x) \geq 4$.
Object $(x_1, x_2, x_3, x_4) = (5, 0, 5, -4)$ satisfies this rule (A direct calculation gives $CPTS(x) = 4$).

- We consider a focal set $T' = \{2,3\}$ of μ^{-c} with $\mu^{-,c}(T') = 5$.
- We consider a focal set $F' = \{1\}$ of μ^{+c} with $\mu^{+c}(F') = 2$.

We obtain the rule
If $x_2^- = x_3^- = 0$ and $x_1^+ \leq 3$ then $0 \leq CPTS_{\mu^-,\mu^+}(x) \leq 3$.
Object $x = (-5, 5, 4, 5)$ satisfies it (a direct calculation gives $CPTS(x) = 3$).

- We consider a focal set $T' = \{3\}$ of μ^{+c} with $\mu^{+c}_{\#}(T') = 5$.
- We consider a focal set $F = \{1\}$ of μ^{-c}, with $\mu^{-c}(F) = 3$

We have the rule
If $x_3^+ = 0$ and $x_1^- \leq 2$ then $0 \leq -CPTS_{\mu^-,\mu^+}(x) \leq 2$.
Object $x = (5, -5, -5, 5)$ satisfies it (a direct calculation gives $CPTS(x) = -2$).

- We consider a focal set $T = \{1,3\}$ of μ^-: with $\mu^-(T) = 3$.
- We consider $F = \{3\}$ is a focal set of $(\mu^+)^c$ with $(\mu^+)^c(F) = 5$.

We have the rule:
if $x_1^- \geq 3$ and $x_3^- \geq 3$ and $x_3^+ = 0$ then $0 \leq -CPTS(x) \leq 3$.
Object $x = (-5, 4, -4, 0)$ satisfies it, a direct calculation gives $CPTS(x) = -3$.
Note that in the last case, since $F \subset T$, the condition $x_3^- \geq 3$ implies $x_3^+ = 0$ so, the latter condition in the rule can be omitted.

5 Conclusion

After recalling the key contribution of Michio Sugeno to the study of non-additive integrals, and the possibility of interpreting Sugeno integrals in terms of threshold if-then rules, we have provided preliminary results on the same question for bipolar Sugeno integrals. In future works we should complete this study by providing conditions under which a set of threshold if-then rules on a bipolar scale can be represented by a CPTS integral. Moreover instead of a bipolar scale, one could use pairs of positive and negative evaluations and compare the worth of decisions by comparing such pairs, as done in [**?**], [**?**],[**?**].

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