A robust interpretation of criteria interactions: Explaining the Human Development Index ranking by a 2-additive Choquet Integral model

Brice Mayag

University Paris-Dauphine
PSL Research university
LAMSADE, CNRS, UMR 7243
Place du Maréchal de Lattre de Tassigny
75775 Paris cedex 16 France
brice.mayag@dauphine.fr

Abstract. The Human Development Index (HDI) is a widely recognized measure, designed to assess the overall well-being and development of nations. This article explores, in the context of Multiple Criteria Decision Analysis, the integration of the discrete 2-additive Choquet Integral into the computation of the HDI, by analyzing the interactions among its dimensions. We show that the HDI formula is equivalent to an additive model and then these interactions can only be interpreted as the possible interactions.

Keywords: MCDA, HDI, Interaction, Choquet integral, Capacity

1 Introduction

The Human Development Index (HDI) was introduced in 1990 by the United Nations Development Programme (UNDP) to move beyond traditional economic indicators and provide a more holistic understanding of a country's development [1, 15]. In this case, it serves as a valuable tool for policymakers, researchers, and the general public to assess a nation's progress beyond solely focusing on economic indicators. HDI is computed based on three essential dimensions: health, education and standard of living. However, a critical evaluation of its computation, formula, illustrative examples, and inherent limitations is essential for a comprehensive understanding of its strengths and weaknesses. As a composite indicator, it can be analyzed as an aggregation function, in a Multiple Criteria Decision Analysis (MCDA) context.

MCDA aims at helping the Decision Maker (DM) to analyze, through a comprehensive mathematical model, his preferences given over a set of alternatives evaluated on a finite set of criteria. The discrete Choquet integral goes beyond traditional averaging MCDA models by considering the interactions and dependencies between these criteria. It appears in [14] as a powerful tool, from fuzzy measure theory, for analyzing the HDI in a more nuanced and flexible manner, in particular the interactions among its three dimensions or criteria. However, all the parameters of this model are fixed by the authors and the obtained rankings are different of those produced by the original HDI.

In this paper, we will focus on countries HDI rankings, by showing that it is always modeled by a Choquet integral w.r.t. a 2-additive capacity, also called the 2-additive Choquet integral. Therefore, through the concepts of possible and necessary interactions recently introduced in [8, 11], we are interested in the robust interpretation of the exisiting interactions among the criteria health, education and standard of living, captured by the interaction indices elaborated from a 2-additive capacity. We highlight that such interpretations should be done in a delicate manner. To do so, we first prove that the geometric mean formula of the HDI is equivalent to an additive model. In this case, the three HDI criteria satisfy the preferential independence property, i.e., there is no interaction taken into account in the HDI model. Secondly, we show that it is always possible to reproduce any ranking of the HDI by using a 2-additive Choquet integral with non null interactions. Hence, we conclude that such interactions can only be interpreted as the possible interactions, but not necessary.

We define and present the computation aspects of the HDI in the next section. Section 3 is devoted to the 2-additive Choquet integral and our results are given in Section 4. The paper ends by a conclusion.

2 Computation of the HDI

The Human Development Index (HDI) is based on three criteria or dimensions: health, education and standard of living [15, 14, 1]. Each of the three dimensions of HDI is individually assessed using specific indicators with established minimum and maximum values. These values are then normalized to a scale of 0 to 1, where 0 represents the lowest level of development and 1 represents the highest. These three criteria are described as follows[1] [14]:

1. *Life expectancy index (LEI)*: the health dimension is assessed by life expectancy at birth (human development longevity)
$$LEI = \frac{\text{life expectancy at birth} - 20}{85 - 20} \tag{1}$$
2. *Educational index (EI)*: The education dimension is measured by mean of years of schooling for adults aged 25 years and more and expected years of schooling for children of school entering age:
$$EI = \frac{MYSI + EYSI}{2} \tag{2}$$
where

[1] https://en.wikipedia.org/wiki/Human_Development_Index

- $MYSI$ refers to the Mean Years of Schooling Index given by

$$MYSI = \frac{\text{MYS}}{15}$$

Fifteen is the projected maximum of this indicator for 2025. MYS being the Mean Years of Schooling (i.e., years that a person aged 25 or older has spent in formal education)

- The Expected Years of Schooling Index (EYSI) is

$$EYSI = \frac{\text{EYS}}{18}$$

Eighteen is equivalent to achieving a master's degree in most countries. Expected Years of Schooling (i.e., total expected years of schooling for children under 18 years of age)

3. *Income Index (II)*: The **standard of living** dimension is measured by gross national income per capita, i.e.,

$$II = \frac{\ln(GNI_{pc}) - \ln(100)}{\ln(75.000) - \ln(100)} \tag{3}$$

where GNI_{pc} is the Gross national income at purchasing power parity per capita.

The original Human Development Index (HDI) formula utilized a simple arithmetic mean to combine these three criteria of human development. While this simple approach has been influential, it presents some limitations that are crucial to consider [13]:

- *Sensitivity to Outliers*: The use of the arithmetic mean is susceptible to the influence of extreme values (outliers) in any of the dimensions. A single dimension with a significantly high or low value can disproportionately affect the overall HDI score.
- *Loss of Information*: The arithmetic mean masks the underlying distribution of values within each dimension. It fails to capture the complete picture of inequalities and disparities within a country.
- *Limited Consideration of Interactions*: The arithmetic mean being an additive model, the three criteria are independent, i.e., they satisfy the preferential independence axiom. Therefore the formula does not take into account potential interactions or synergies between the different dimensions. For example, a strong education system might have a positive impact on health outcomes, but this relationship is not directly reflected in the simple average.

In recognition of these limitations, the UNDP introduced a revised HDI formula in 2010. This new version utilizes a geometric mean, which is less sensitive to outliers and provides a more balanced representation of development across dimensions. Additionally, the concept of Inequality-adjusted Human Development

Index (IHDI) was introduced, acknowledging the importance of addressing inequalities within countries [7, 13]. For a country x, the HDI of x is given by the following expression:

$$HDI(x) = \sqrt[3]{LEI(x) \times EI(x) \times II(x)} \tag{4}$$

The ranking of the first ten and last ten countries is given by Figure 1 where we added a column giving the value of the geometric mean. We noticed that this latter is different from the real HDI value, for eight of the top ten countries (shown in bold).

Example 1 *In 2021, Switzerland was the best country in the HDI's ranking. Its HDI score is computed as follows:*

- $LEI(Switzerland) = (84 - 20)/(85 - 20) = 0.9844$*;*
- $EI(Switzerland) = ((16.5/18) + (13.9/15))/2 = 0.9203$*;*
- $II(Switzerland) = (\ln(66933) - \ln(100))/\ln(75.000) - \ln(100) = 0.9828$*;*
- $HDI(Switzerland) = \sqrt[3]{0.9844 \times 0.9203 \times 0.9828} = 0.962$*.*

HDI rank 2021	Country	Human Development Index (HDI) real value	HDI Geometric formula	Life expectancy at birth (years)	Expected years of schooling (years)	Mean years of schooling (years)	Gross national income (GNI) per capita	LEI	EI	II
1	Switzerland	0,962	0,96205024	84,0	16,5	13,9	66 933	0,9844185	0,92033	0,98281
2	**Norway**	**0,961**	**0,962848706**	83,2	18,2	13,0	64 660	0,9728292	0,938599	0,977592
3	**Iceland**	**0,959**	**0,970138869**	82,7	19,2	13,8	55 782	0,96428	0,991213	0,955282
4	**Hong Kong, China (SAR**	**0,952**	**0,954480653**	85,5	17,3	12,2	62 607	1,0072831	0,887489	0,972717
5	**Australia**	**0,951**	**0,978908363**	84,5	21,1	12,7	49 238	0,9927154	1,009077	0,936434
6	**Denmark**	**0,948**	**0,954390904**	81,4	18,7	13,0	60 365	0,9442354	0,951872	0,967208
7	**Sweden**	**0,947**	**0,960121456**	83,0	19,4	12,6	54 489	0,9689738	0,959728	0,95174
8	**Ireland**	**0,945**	**0,955452302**	82,0	18,9	11,6	76 169	0,9538092	0,91233	1,002336
9	Germany	0,942	0,942247143	80,6	17,0	14,1	54 534	0,9327708	0,942203	0,951865
10	**Netherlands**	**0,941**	**0,947804722**	81,7	18,7	12,6	55 979	0,9490354	0,938642	0,955816
182	Guinea	0,465	0,464709871	58,9	9,8	2,2	2 481	0,5983415	0,345774	0,48507
183	Yemen	0,455	0,454899895	63,8	9,1	3,2	1 314	0,6731292	0,359409	0,389099
184	Burkina Faso	0,449	0,448797756	59,3	9,1	2,1	2 118	0,6041477	0,324446	0,461176
185	Mozambique	0,446	0,445823099	59,3	10,2	3,2	1 198	0,6049954	0,390453	0,375117
186	Mali	0,428	0,428033988	58,9	7,4	2,3	2 133	0,5990985	0,283195	0,462222
187	Burundi	0,426	0,426348843	61,7	10,7	3,1	732	0,6409646	0,402162	0,300649
188	Central African Republic	0,404	0,403529682	53,9	8,0	4,3	966	0,5214569	0,367805	0,342603
189	Niger	0,400	0,400358262	61,6	7,0	2,1	1 240	0,6396354	0,26381	0,380296
190	Chad	0,394	0,393726351	52,5	8,0	2,6	1 364	0,5003908	0,309012	0,394728
191	South Sudan	0,385	0,385143027	55,0	5,5	5,7	768	0,53808	0,34483	0,307903

Fig. 1. Top 10 and last 10 countries of the HDI ranking (2021)

3 The discrete 2-additive Choquet integral

3.1 Definitions

The discrete 2-additive Choquet Integral, rooted in fuzzy set theory, extends the traditional Choquet Integral by considering the interactions between pairs of criteria. This section introduces the mathematical foundations of this aggregation function, highlighting its capacity to capture interdependencies.

Let $X = X_1 \times \ldots \times X_n$ be a finite set of alternatives evaluated on a set of n criteria $N = \{1, \ldots, n\}$. X is also viewed as Cartesian product of the n attributes $X_1, \ldots, X_n$. We denote by 2^N the set of all subsets of N. The 2-additive Choquet integral [9, 10] is an aggregation function based on a *capacity or fuzzy measure* μ, defined as a set function from 2^N to $[0, 1]$ such that:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$
3. $\forall A, B \in 2^N,\ [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

The *Möbius transform* $m^\mu : 2^N \to \mathbb{R}$ associated to the capacity μ is defined by

$$m^\mu(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K), \forall T \in 2^N. \tag{5}$$

while a capacity μ on N is said to be *2-additive* if it satisfies the following two conditions:

- For all subset T of N such that $|T| > 2$, $m^\mu(T) = 0$;
- There exists a subset B of N such that $|B| = 2$ and $m^\mu(B) \neq 0$.

Given a capacity μ and its Möbius transform m^μ, we adopt, in the sequel, the notations $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i, j\})$, $m_i^\mu := m^\mu(\{i\})$, $m_{ij}^\mu := m^\mu(\{i, j\})$, for all $i, j \in N$, $i \neq j$. Whenever we use i and j together, it always means that they are different.

For an alternative $x := (x_1, ..., x_n)$ of X, the 2-additive Choquet integral of x is expressed as follows [6]:

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^\mu\ u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu\ |u_i(x_i) - u_j(x_j)| \tag{6}$$

where

- For all $i \in N$, $u_i : X_i \to \mathbb{R}_+$ is a marginal utility function associated to the attribute X_i;
- $I_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$ is the interaction index between the two criteria i and j [5, 12];
- $\phi_i^\mu = \sum_{K \subseteq N \setminus i} \frac{(n - |K| - 1)!|K|!}{n!} (\mu(K \cup i) - \mu(K)) = \mu_i + \frac{1}{2} \sum_{j \in N, j \neq i} I_{ij}^\mu$ is defined as the importance of criterion i and it corresponds to the Shapley value of i w.r.t. μ [16].

Equation (6) is equivalent to the following Equation (7), related to the coefficients of the Möbius transform of μ:

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i \in N} m_i^\mu\ u_i(x_i) + \sum_{i,j \in N} m^\mu(\{i, j\})\ \min(u_i(x_i), u_j(x_j)) \tag{7}$$

The interaction index $I_{ij}^{\mu} = m^{\mu}(\{i,j\}) = \mu_{ij} - \mu_i - \mu_j$ is usually interpreted as follows:

- A positive interaction index ($I_{ij}^{\mu} > 0$) indicates complementarity between criteria. This implies that the combined effect of two criteria is greater than the sum of their individual effects. In other words, having high values in both criteria leads to a greater overall score than one would expect by simply adding their individual contributions.

  For instance, the criteria LEI and EI may have a positive interaction. Indeed a high education paired with strong healthcare systems can lead to a greater increase in life expectancy than the sum of the benefits from each alone.
- A negative interaction index ($I_{ij}^{\mu} < 0$) suggests substitutability between criteria. This signifies that the combined effect of two criteria is less than the sum of their individual effects. In simpler terms, having a high value in one dimension can partially compensate for a lower value in another criterion, leading to an overall score that is not as negatively impacted as one might expect.

  In the context of HDI, the criteria EI and II could interact negatively. Indeed, a strong economy might partially offset the negative effects of low education levels on certain aspects of human development.
- A null interaction index ($I_{ij}^{\mu} = 0$) implies that the criteria are independent, meaning their combined effect is simply the sum of their individual effects.

In general, the interpretation of I_{ij}^{μ} depends on the specific context and the values assigned to the capacity μ. Therefore, to better interpret such index and then have robust interpretations, the following notion of possible and necessary interactions have been recently introduced in MCDA, especially in preference modeling [8, 11].

3.2 Necessary and possible interactions

Let us assume that the DM is able to express his preferences on X through a strict preference relation P and an indifference relation I on X. The preference information $\{P, I\}$ is an ordinal information which is representable by a 2-additive Choquet integral if there exists a capacity μ such that for all $x, y \in X$,

$$\begin{cases} x \; P \; y \Longrightarrow C_\mu(u_1(x_1), \ldots, u_n(x_n)) > C_\mu(u_1(y_1), \ldots, u_n(y_n)) \\ x \; I \; y \Longrightarrow C_\mu(u_1(x_1), \ldots, u_n(x_n)) = C_\mu(u_1(y_1), \ldots, u_n(y_n)) \end{cases} \quad (8)$$

Let us consider $\mathcal{C}_{pref}$ the set of all capacities μ such that a preference information $\{P, I\}$ is representable by a 2-additive Choquet integral C_μ. Let be $i, j \in N$, $i \neq j$, two criteria.

1. There exists a *possible* positive (respectively, null, negative) interaction between i and j if there exists a capacity $\mu \in \mathcal{C}_{pref}$ such that $I_{ij}^{\mu} > 0$ (respectively, $I_{ij}^{\mu} = 0$, $I_{ij}^{\mu} < 0$).

2. There exists a *necessary* positive (respectively, null, negative) interaction between i and j if $I^{\mu}_{ij} > 0$ (respectively, $I^{\mu}_{ij} = 0$, $I^{\mu}_{ij} < 0$) for all capacities $\mu \in \mathcal{C}_{pref}$.

Many characterizations related to the existence of necessary and possible interactions can be founded in [8, 11]. We just give here a result about the null interaction, since we need it in the next section.

Proposition 1 *Let $\{P, I\}$ a preference information on X representable by a 2-additive Choquet integral.*

If $I = \emptyset$ then there is no necessary null interactions.

Proof. See [11], Proposition 1, Page 4. This proof is constructive, i.e., it provides a way to elaborate from a null interaction $I^{\mu}_{ij} = 0$, another 2-additive Choquet integral model C_{ν} where this $I^{\nu}_{ij} > 0$.

This result shows that if there exists a representation of a strict preference P by C_{μ} such that $I^{\mu}_{ij} = 0$, then it is always possible to get another representation of P such that $I^{\mu}_{ij} \neq 0$ (in fact $I^{\mu}_{ij} > 0$). In particular, it is true if P is a linear order on the set of alternatives (a ranking with no ties), like a linear order induced by a ranking of the countries given by the HDI.

4 Our results

4.1 An additive model equivalent to the HDI

Let x be a country evaluated on the three criteria LEI, EI and II. As described in Section 2, the values $LEI(x)$, $EI(x)$ and $II(x)$ belong to $[0, 1]$. Since we will use the logarithmic function in the rest of the paper, w.l.o.g., we can assume all these three values belong to $]0, 1]$ as the HDI value of any country is never equals to zero.

Let us define the aggregation function $\widetilde{HDI}$ assigning to x a global score $\widetilde{HDI}(x)$ as follows:

$$\widetilde{HDI}(x) = \frac{\ln(1000 \times LEI(x))}{3} + \frac{\ln(1000 \times EI(x))}{3} + \frac{\ln(1000 \times II(x))}{3} \quad (9)$$

It is clear that $\widetilde{HDI}$ is an additive value function of three criteria, where the marginal utility function associated to the criteria $i \in \{LEI, EI, II\}$, is given by $x \to \ln(1000 \times x)$. Therefore the aggregation function $\widetilde{HDI}$ satisfies the preferential independence axiom [2–4], i.e., the criteria LEI, EI and II are independent[2]. Proposition 2 below provides a link between the two aggregation functions HDI and $\widetilde{HDI}$:

[2] An attribute is preferentially independent from all other attributes when changes in the rank ordering of preferences of other attributes does not change the preference order of the attribute.

Proposition 2 *Given two countries x and y, we have*

$$HDI(x) > HDI(y) \iff \widetilde{HDI}(x) > \widetilde{HDI}(y) \tag{10}$$

$$HDI(x) = HDI(y) \iff \widetilde{HDI}(x) = \widetilde{HDI}(y) \tag{11}$$

In other words, the rankings of the countries, obtained by HDI and $\widetilde{HDI}$, are identical.

Proof. The proof is based on the fact ln is an increasing function, i.e., $\forall a, b \in]0,1]$, $a > b \Leftrightarrow \ln(a) > \ln(b)$ and $a = b \Leftrightarrow \ln(a) = \ln(b)$.

Let x, y be two countries. We have:

$$\begin{aligned}
& HDI(x) > HDI(y) \\
\Leftrightarrow\ & (LEI(x) \times EI(x) \times II(x))^{1/3} > (LEI(y) \times EI(y) \times II(y))^{1/3} \\
\Leftrightarrow\ & 1000 \times (LEI(x) \times EI(x) \times II(x))^{1/3} > 1000 \times (LEI(y) \times EI(y) \times II(y))^{1/3} \\
\Leftrightarrow\ & \sum_{i \in \{LEI, EI, II\}} \ln(1000 \times i(x)) > \sum_{i \in \{LEI, EI, II\}} \ln(1000 \times i(y)) \\
\Leftrightarrow\ & \widetilde{HDI}(x) > \widetilde{HDI}(y)
\end{aligned}$$

The same reasoning is applied to $HDI(x) = HDI(y) \iff \widetilde{HDI}(x) = \widetilde{HDI}(y)$.

Figure 2 illustrates the rankings of the countries obtained from HDI and $\widetilde{HDI}$, given the evaluations of the twenty countries of Figure 1.

Country	HDI value Geometric mean formula	HDI~ value Logarithmic formula	LEI	EI	II	Ln(1000*LEI)	Ln(1000*EI)	Ln(1000*II)
Australia	0,978908363	6,886438035	0,9927154	1,0090771	0,9364337	6,900444001	6,916791392	6,84207871
Iceland	0,970138869	6,877439225	0,96428	0,9912129	0,9552821	6,871381709	6,89892931	6,86200666
Norway	0,962848706	6,869896292	0,9728292	0,9385988	0,9775918	6,880208559	6,844388082	6,88509224
Switzerland	0,96205024	6,869066674	0,9844185	0,9203303	0,9828105	6,892051072	6,824732652	6,8904163
Sweden	0,960121456	6,867059793	0,9689738	0,9597276	0,9517404	6,876237621	6,8666495	6,85829226
Ireland	0,955452302	6,862184843	0,9538092	0,9123302	1,0023363	6,860463684	6,816002028	6,91008882
Hong Kong, China (SAR)	0,954480653	6,861167373	1,0072831	0,8874895	0,9727173	6,915011962	6,788396665	6,88009349
Denmark	0,954390904	6,86107334	0,9442354	0,9518719	0,9672085	6,850375483	6,85843046	6,87441408
Netherlands	0,947804722	6,854148492	0,9490354	0,9386422	0,9558156	6,855446084	6,844434414	6,86256498
Germany	0,942247143	6,8482676	0,9327708	0,9422028	0,9518646	6,838159479	6,848220507	6,85842281
Guinea	0,464709871	6,141413278	0,5983415	0,3457736	0,4850704	6,394161725	5,845784146	6,18429396
Yemen	0,454899895	6,120077383	0,6731292	0,3594086	0,3890994	6,511937333	5,884459938	5,96383488
Burkina Faso	0,448797756	6,106572355	0,6041477	0,324446	0,4611758	6,403818692	5,782119031	6,13377934
Mozambique	0,445823099	6,099922235	0,6049954	0,3904534	0,3751167	6,405220829	5,967308586	5,92723729
Mali	0,428033988	6,059202604	0,5990985	0,2831955	0,4622216	6,395425961	5,646137434	6,13604442
Burundi	0,426348843	6,055257892	0,6409646	0,4021623	0,3006491	6,462974253	5,996855694	5,70594373
Central African Republic	0,403529682	6,000250046	0,5214569	0,3678048	0,3426026	6,256626669	5,907552281	5,83657119
Niger	0,400358262	5,992359801	0,6396354	0,2638104	0,3802963	6,460898302	5,575230489	5,94095061
Chad	0,393726351	5,975656126	0,5003908	0,3090123	0,3947284	6,215389332	5,733381135	5,97819791
South Sudan	0,385143027	5,953614765	0,53808	0,3448299	0,3079033	6,288007248	5,843051383	5,72978566

Fig. 2. The rankings of HDI and $\widetilde{HDI}$ obtained by using data given in Figure 1

Remark 1

- *Due to some drawbacks given above (see Section 2), for instance the limitation about the fully compensation between the three dimensions, the old formula of the HDI (arithmetic mean) was changed into the geometric mean. Since we proved here that this latter is still equivalent to an arithmetic mean (by using an appropriate normalization), we think that the previous limitations remain valid in this geometric mean model.*
- *In fact, $\widetilde{HDI} = \phi(HDI)$ with strictly monotonous ϕ, which implies they lead to the same ranking.*

In the next section, we show that the positive or negative interactions among the HDI dimensions are never necessary, when the obtained HDI ranking is modeled by a 2-additive Choquet integral .

4.2 About the possible and necessary interactions among the HDI dimensions

Let us denoted by $R(HDI)$ the linear order induced by the ranking of countries given by the HDI. In this case, the countries with the same HDI value are considered as a single country. The rankings of HDI and $\widetilde{HDI}$ being identical, we will also have $R(HDI)$ and $R(\widetilde{HDI})$ identical . Therefore each of these linear orders of the evaluated countries could be assimilated to a strict preference information.

Proposition 3 *It is always possible to represent $R(HDI)$ by a 2-additive Choquet integral C_μ model with non null interactions.*

Proof. This result is a consequence of the Proposition 1. Indeed, one can notice that $R(HDI)$ is a strict preference information obtained from an additive model $\widetilde{HDI}$, i.e., a 2-additive Choquet integral model where there is no interaction ($I^\mu_{ij} = 0$, for $i, j \in \{LEI, EI, II\}$). In this case, such null interactions are not necessary, i.e., it is possible to represent $R(HDI)$ by using a 2-additive Choquet integral where all the interaction indices are not null. This leads to have two representations of the same ranking, one with a null interaction $I^\mu_{ij} = 0$ and the other one with non null interaction $I^\nu_{ij} \neq 0$. Hence, such interactions are never necessary. They can only be interpreted in a possible way.

Example 2 *In order to illustrate the previous result and to show how to build a 2-additive capacity from the additive model $\widetilde{HDI}$, we adopt the procedure given in the original proof of Proposition 1 (See [11], Proposition 1, Page 4.). We consider the data of the twenty countries given in Figure 2.*

If we set, for our convenience, $1 := LEI$, $2 := EI$ and $3 := II$, then the additive capacity μ associated to the $\widetilde{HDI}$, of these twenty countries, is $\mu_1 = \mu_2 = \mu_3 = 1/3$, $\mu_{12} = \mu_{13} = \mu_{23} = 2/3$. We can notice that $I^\mu_{ij} = 0$ for all $i, j \in \{1, 2, 3\}$. The following steps allow us to compute a model C_ν such that $I^\nu_{12} > 0$:

(i) Compute α the smallest difference between the value of $\widetilde{HDI}$ for two consecutive countries in the given ranking. We get $\alpha = 0.0000940330483656382$.

(ii) For each country x, determine $k_{12}^x = \min(LEI(x), EI(x))$, $LEI(x)$ and $EI(x)$ being the marginal utility functions used in $\widetilde{HDI}$.

(iii) Determine $k_{12} = \min_x(k_{12}^x)$. We get $k_{12} = 5.57523048923919$.

(iv) Determine a value $\epsilon > 0$ such that $\epsilon \times k_{12} < \alpha$ and set $I_{12}^{\nu} = \nu_{12} - \nu_1 - \nu_2 = \epsilon$. We choose $\epsilon = 0.000015$

(v) Using an appropriate normalization, build a 2-additive capacity ν that has the same Möbius transform as μ, but has $I_{12}^{\nu} = \nu_{12} - \nu_1 - \nu_2 = \epsilon$. We get $\nu_1 = \mu_1/(1+\epsilon) = \nu_2 = \nu_3 = 0.333328333408332$, $\nu_{12} = \epsilon + \nu_1 + \nu_2 = 0.666671667$ and $\nu_{13} = \nu_{23} = 0.666656666816664$.

(vi) The ranking of the twenty countries based on C_ν, with $I_{12}^{\nu} > 0$ is given in Figure 3.

Country	HDI value Geometric mean formula	HDI~ value Logarithmic formula (no interaction)	2-additive Choquet integral (LEI and EI interact)
Australia	0,978908363	6,886438035	6,886334495
Iceland	0,970138869	6,877439225	6,877335652
Norway	0,962848706	6,869896292	6,869792708
Switzerland	0,96205024	6,869066674	6,86896263
Sweden	0,960121456	6,867059793	6,866956645
Ireland	0,955452302	6,862184843	6,862081245
Hong Kong, China (SAR)	0,954480653	6,861167373	6,861062558
Denmark	0,954390904	6,86107334	6,860970305
Netherlands	0,947804722	6,854148492	6,854045516
Germany	0,942247143	6,8482676	6,848164726
Guinea	0,464709871	6,141413278	6,141312932
Yemen	0,454899895	6,120077383	6,119976171
Burkina Faso	0,448797756	6,106572355	6,106471432
Mozambique	0,445823099	6,099922235	6,099824169
Mali	0,428033988	6,059202604	6,059100478
Burundi	0,426348843	6,055257892	6,055160073
Central African Republic	0,403529682	6,000250046	6,000154807
Niger	0,400358262	5,992359801	5,992256632
Chad	0,393726351	5,975656126	5,975559263
South Sudan	0,385143027	5,953614765	5,953518788

Fig. 3. The rankings of HDI obtained by using the 2-additive capacity ν

We conclude that a representation of a HDI ranking by a 2-additive Choquet integral model, with non null interactions, always exists, even if it is proved that the criteria LEI, EI and II are independent since $\widetilde{HDI}$ is an additive model. Therefore the interpretation of these interactions is not robust and should be in caution. Clearly, in this context, the following equivalence is false :

$$I_{ij}^{\mu} \neq 0, \text{ for } i,j \in \{LEI, EI, II\} \Longleftrightarrow i,j \text{ are not independent}$$

5 Conclusion

We showed that the 2-additive Choquet integral presents a promising approach for analyzing the interactions between the dimensions of the HDI, especially when we need to explain the ranking of the countries provided by this composite indicator. But the interpretation of these interactions is limited, since they are never necessary, i.e., their sign strongly depends on the values of the capacity.

In the future work, we will investigate how to elaborate a 2-additive Choquet integral model for HDI incorporating necessary interactions between the pairs of the dimensions. An idea could be to ask a preference information from a set of binary alternatives (fictitious countries with only 0-1 values in each criterion). As these alternatives are directly related to the interaction index, they could be useful to obtain a new HDI model allowing to have robust interpretations of the interactions.

References

1. S. Ambuj and N. Adil. The human development index: A critical review. *Ecological Economics*, 25:249–264, 06 1998.
2. D. Bouyssou, T. Marchant, M. Pirlot, P. Perny, A. Tsoukiàs, and Ph. Vincke. *Evaluation and decision models: a critical perspective*. Kluwer Academic, Dordrecht, 2000.
3. Denis Bouyssou, Thierry Marchant, Marc Pirlot, Alexis Tsoukiàs, and Philippe Vincke. *Evaluation and decision models with multiple criteria: Stepping stones for the analyst*. International Series in Operations Research and Management Science, Volume 86. Boston, 1st edition, 2006.
4. J.S. Dyer. MAUT - multiattribute utility theory. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 265–285. Springer Verlag, Boston, Dordrecht, London, 2005.
5. M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.
6. M. Grabisch and Ch. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *4OR*, 6:1–44, 2008.
7. Salvatore Greco, Alessio Ishizaka, Menelaos Tasiou, and Gianpiero Torrisi. On the methodological framework of composite indices: A review of the issues of weighting, aggregation, and robustness. *Social Indicators Research*, 141(1):61—-94, 2019.
8. Paul Alain Kaldjob Kaldjob, Brice Mayag, and Denis Bouyssou. On the interpretation of the interaction index between criteria in a Choquet integral model. *Fuzzy Sets and Systems*, 458:165–181, 2023.
9. B. Mayag, M. Grabisch, and C. Labreuche. A characterization of the 2-additive Choquet integral through cardinal information. *Fuzzy Sets and Systems*, 184(1):84–105, 2011.
10. B. Mayag, M. Grabisch, and C. Labreuche. A representation of preferences by the Choquet integral with respect to a 2-additive capacity. *Theory and Decision*, 71(3):297–324, 2011.
11. Brice Mayag and Denis Bouyssou. Necessary and possible interaction between criteria in a 2-additive Choquet integral model. *European Journal of Operational Research*, 283(1):308 – 320, 2020.

12. T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.
13. Riccardo Natoli, Simon Feeny, Junde Li, and Segu Zuhair. On the methodological framework of composite indices: A review of the issues of weighting, aggregation, and robustness. *Social Indicators Research*, 2024.
14. Mehmet Pinar. Choquet-integral aggregation method to aggregate social indicators to account for interactions: an application to the human development index. *Social Indicators Research*, 159(1):1–53, 2022.
15. Gustav Ranis, Frances Stewart, and Emma Samman. Human development: Beyond the human development index. *Journal of Human Development*, 7(3):323–358, 2006.
16. L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.