Bipolar fuzzy extension of topological structures on fuzzy powersets

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Abstract. We present a model allowing to extend an L-topology τ on a set X (i.e. $\tau \subseteq L^X$) to a bipolar $\mathcal{L}$-fuzzy topology $\mathcal{T}$ on this set (i.e. $\mathcal{T} : L^X \to \mathcal{L}$). This model is based on the use of a residuated-type structure on a lattice L, and the derived lattice $\mathcal{L}$ obtained by "bipolarizing" the original lattice L. The properties of the obtained bipolar $\mathcal{L}$-fuzzy topology are studied. We specially consider the case when the original lattice L is enriched with a monoid Girard structure. In this case, the results obtained become especially transparent.

Keywords: Complete infinitely distributive lattices, fuzzy topologies, (L, M)-fuzzy topologies, bipolar lattices

1 Introduction

In "classic" (non-fuzzy) mathematics it usually seems pointless to ask to what extent a given object has a certain property: a topological space is either compact or not, a metric space is either complete or not, a group is either commutative or not, et al. On the other hand, within the framework of fuzzy mathematical structures, the tools of "fuzzy logic" make it possible to give some meaning to this issue. In particular, a quite much work was done to estimate to what degree a fuzzy topological space or its fuzzy subset is compact, Hausdorff, connected et al., to what degree a function of (fuzzy) topological spaces is continuous et al., see, for example [23], [25], [28], [33], [34], [35] et al. Such an assessment of the presence of a property is usually evaluated by a value in a complete lattice L. However, in certain situations it may be appropriate to combine the assessment of the presence of a property with the assessment of the presence of the opposite property in a given object. As a tool for implementing this approach, more general assessment scales can be used, in particular, those based on the so-called "intuitionistic" fuzzy sets or on bipolar-valued fuzzy sets. The purpose of this work is to initiate the use of bipolar fuzzy estimation of specific topological properties within the framework of fuzzy mathematical structures. Namely, in this paper we present a bipolar fuzzy estimation of the properties of openness and closedness for fuzzy sets in a fuzzy topological space and, on this basis, develop a model that allows us to extend an L-fuzzy topology τ on a set X to the fuzzy topology $\mathcal{T} : L^X \to \mathcal{L}$, where $\mathcal{L}$ is the bipolar extension of the original lattice L. Along with the purely theoretical interest in this model, we assume that it can serve as a tool for a deeper (compared to previously undertaken) analysis of (fuzzy) topologies and their properties, in particular compactness, connectivity and separation properties of fuzzy topological spaces.

2 Preliminary information: the framework of our study

2.1 Construction of the bipolar lattice $\mathcal{L}$

Quantales and residuated lattices We use the standard terminology accepted in theory of lattices, see, e.g. [3], [11], [30]. When speaking about a lattice $(L, \leq, \wedge, \vee)$, we assume that it is complete with $\mathbf{0}_L$ and $\mathbf{1}_L$ its bottom and top elements respectively. A frame, or an infinitely distributive lattice is a complete lattice satisfying the infinite distributivity law

$$(\bigvee_{i\in I} a_i) \wedge b = \bigvee_{i\in I}(a_i \wedge b) \ \forall b \in L \ \forall \{a_i \mid i \in I\} \subseteq L.$$

A tuple $(L, \leq, \wedge, \vee, *)$ is called a quantale [32] if $(L, \leq, \wedge, \vee)$ is a complete lattice and $* : L \times L \to L$ a binary associative monotone operation which distributes over arbitrary joins:

$$a * \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a * b_i) \text{ and } \left(\bigvee_{i\in I} b_i\right) * a = \bigvee_{i\in I}(b_i * a)$$

for all $\{b_i \mid i \in I\} \subseteq L$ and for all $a \in L$. A quantale $(L, \leq, \wedge, \vee, *)$ is called (1) symmetric if $a * b = b * a$ for every $a, b \in L$, (2) integral if $1_L \in L$ is the unit element of the monoid $(L, \leq, \wedge, \vee, *)$. When saying *a quantale*, we always assume that it is symmetric and integral.

In a quantale a further binary operation $\mapsto: L \times L \to L$, the residuum, can be introduced as associated with operation $*$ via the Galois connection:

$$a * b \leq c \Longleftrightarrow a \leq b \mapsto c \ \text{ for all } \ a, b, c \in L.$$

A quantale $(L, \leq, \wedge, \vee, *)$ provided with the derived operation $\mapsto$, that is the tuple $(L, \leq, \wedge, \vee, *, \mapsto)$, is known also as a (complete) residuated lattice [30]. In what follows we usually call operation $*$ the conjunction, and the corresponding residuum $\mapsto$ the implication in the residuated lattice L.

Complementation A unary relation $^c : L \to L$ is called a complementation if it is an order reversing involution that is

$$(\neg_1)\ a \leq b \Longrightarrow b^c \leq a^c \text{ for all } a, b \in L; \ (\neg_2)\ (a^c)^c = a \text{ for every } a \in L.$$

A typical example of complementation is the subtraction operation on the unit interval, that is $a^c = 1 - a$ for every $a \in [0, 1]$. Another important example is complsmentation defined by means of residuum $a^c = a \mapsto 0$ in a Girard monoid. The following property of complementation is well known.

Proposition 1. *A complementation in a completele lattice satisfies the generalized de Morgan laws:*

$$\left(\bigvee_{i\in I} a_i\right)^c = \bigwedge_{i\in I} a_i^c \text{ and } \left(\bigwedge_{i\in I} a_i\right)^c = \bigvee_{i\in I} a_i^c \text{ for all } \{a_i \mid i \in I\} \subseteq L.$$

Construction of the bipolar lattice $\mathcal{L}$ from a complete lattice L We use two copies of the lattice L denoted respectively by L^+ and L^-. The elements of $L = L^+$ are denoted $\mathbf{0}, a, \mathbf{1}$, while the corresponding elements of L^- are denoted $\mathbf{0}^-, a^-, \mathbf{1}^-$, respectively. We consider correspondence

$$^\sim : L^+ \longleftrightarrow L^- \text{ defined by } {}^\sim a = a^- \text{ and } {}^\sim a^- = a$$

and introduce the order $\leq^-$ on L^- by setting $b_1^- \leq^- b_2^- \iff b_1 \geq b_2$. So, actually the lattice L^- can be defined as the lattice L^{op}. The supremum and the infimum on the lattice L^- are defined by the infimum and supremum on the lattice L, i.e.

$$\bigvee_{i\in I}^{L^-} b_i = \bigwedge_{i\in I}^{L} {}^\sim b_i \text{ and } \bigwedge_{i\in I}^{L^-} b_i = \bigvee_{i\in I}^{L} {}^\sim b_i \ \forall \{b_i \mid i \in I\} \subseteq L^-.$$

Further, let $\mathcal{L} = L^+ \times L^-$. We introduce a partial order $\preceq$ on $\mathcal{L}$ by setting

$$(a, b) \preceq (a', b') \iff a \leq a', b \geq b'.$$

Given a family $\mathcal{F} = \{(a_i, b_i) \mid\in I\} \subseteq \mathcal{L}$, we define

$$\bigvee \mathcal{F} = \left(\bigvee_{i\in I}^{L} a_i \bigvee_{i\in I}^{L^-} b_i\right) \quad \text{and} \quad \bigwedge \mathcal{F} = \left(\bigwedge_{i\in I}^{L} a_i, \bigwedge_{i\in I}^{L^-} b_i\right).$$

From the above construction one can easily get the following result:

Theorem 1. *$(\mathcal{L}, \preceq, \bigwedge, \bigvee)$ is a complete lattice. Its top element $\top_{\mathcal{L}}$ is $(\mathbf{1}, \mathbf{0}^-) \in \mathcal{L}$; its bottom element $\bot_{\mathcal{L}}$ is $(\mathbf{0}, \mathbf{1}^-) \in \mathcal{L}$. Moreover, if L is a frame, the the lattice $\mathcal{L}$ is also a frame.*

2.2 Measures of inclusion and non-inclusion on L-powersets

Measure of inclusion The following definition was first introduced in [2] and [37]. Later such definition and its different modification were used by many authors, see, e.g. [9], [19], [33] et al.

Definition 1. *Let X be a set and $(L, \leq, \wedge, \vee, *, \mapsto)$ be a residuated lattice. The measure of inclusion of an L-fuzzy set $A \in L^X$ into an L-fuzzy set $B \in L^X$ is defined by $A \hookrightarrow B = \bigwedge_{x\in X} A(x) \mapsto B(x)$.*

In the following proposition we collect properties of the operation $\hookrightarrow: L^X \times L^X \to L$. The proofs easily follow from the properties of the residuum and can be found in many articles, see, e.g. [13].

Proposition 2. *Let $\{A_i \mid i \in I\} \subseteq L^X$, $\{B_i \mid i \in I\} \subseteq L^X$, $A, B \in L^X$. Then*
(1) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$;
(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$;
(3) $A \hookrightarrow B = \mathbf{1}$ *whenever* $A \leq B$;
(4) $A_1 \leq A_2 \Longrightarrow A_1 \hookrightarrow B \geq A_2 \hookrightarrow B$;
(5) $B_1 \leq B_2 \Longrightarrow A \hookrightarrow B_1 \leq A \hookrightarrow B_2$;
(6) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$;
(7) $(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$.

Measure of non-inclusion We define measure of non-inclusion on the powerset L^X based on the operation $\nrightarrow$ on the lattice L. This operation can be viewed as a certain co-implication.

Definition 2. *Let $a, b \in L$ and let $^c : L \to L$ be a complementation. We define co-implication $\nrightarrow L \times L \to L$ by setting $a \nrightarrow b = a * b^c$.*

Directly from the definition it is easy to establish the basic properties of co-implication collected in the next proposition.

Proposition 3. *Let $a, b \in L$ and $\{a_i \mid i \in I\}$, $\{b_i \mid i \in I\} \subseteq L$. Then:*
(1) $a_1 \leq a_2 \Longrightarrow a_1 \nrightarrow b \leq a_2 \nrightarrow b$;
(2) $b_1 \leq b_2 \Longrightarrow a \nrightarrow b_1 \geq a \nrightarrow b_2$;
(3) $\left(\bigvee_{i\in I} a_i\right) \nrightarrow \left(\bigvee_{i\in I} b_i\right) \leq \bigvee_{i\in I}(a_i \nrightarrow b_i)$;
(4) $\left(\bigwedge_{i\in I} a_i\right) \nrightarrow \left(\bigwedge_{i\in I} b_i\right) \leq \bigvee_{i\in I}(a_i \nrightarrow b_i)$

Basing on the operator $\nrightarrow : L \times L \to L$, we introduce the measure of non-inclusion of one fuzzy set into another:

Definition 3. *Let $A, B \in L^X$. Then the measure of non-inclusion of an L-fuzzy set A into an L-fuzzy set B is defined by $A \not\hookrightarrow B = \bigvee_{x\in X}(A(x) \nrightarrow B(x))$.*

From Proposition 3 we easily prove the following statement collecting the basic properties of the non-inclusion relation.

Proposition 4. *Let $A, B \in L^X$, $\{A_i \mid i \in I\}$, $\{B_i \mid i \in I\} \subseteq L\cdot$ Then*
(1) $A_1 \leq A_2 \Longrightarrow A_1 \not\hookrightarrow B \leq A_1 \not\hookrightarrow B$;
(2) $B_1 \leq B_2 \Longrightarrow A \not\hookrightarrow B_1 \geq A \not\hookrightarrow B_2$;
(3) $\left(\bigvee_{i\in I} A_i\right) \not\hookrightarrow \left(\bigvee_{i\in I} B_i\right) \leq \bigvee_{i\in I}(A_i \not\hookrightarrow B_i)$;
(4) $\left(\bigwedge_{i\in I} A_i\right) \not\hookrightarrow \left(\bigwedge_{i\in I} B_i\right) \leq \bigvee_{i\in I}(A_i \not\hookrightarrow B_i)$.

2.3 Topological structures in the context of fuzzy sets: introductory notes

Fuzzy topologies The first definition of a topology in the context of L-fuzzy sets was introduced by C.L. Chang [7] in case $L = [0, 1]$ and extended for an arbitrary infinitely distributive lattice (frame) by J.A. Goguen [12]. Now following, e.g.[16], [17] such "topologies" are usually referred to as L-topologies in order to emphasize that only sets are L-fuzzy, but the topology itself is still crisp.

Definition 4. [7], [12] *Let X be a set and L be a frame. A family of L-fuzzy subsets $\tau \subseteq L^X$ is called an L-topology on a set X if the following conditions are satisfied: (1) $\mathbf{0}_X, \mathbf{1}_X \in \tau$; (2) $A, B \in \tau \Longrightarrow A \wedge B \in \tau$; (3) $\{A_i \mid i \in I\} \subseteq L \Longrightarrow \bigvee_{i\in I} A_i \in \tau$. The corresponding pair (X, τ) is called an L-topological space. Given two L-topological spaces (X, τ_X) and (Y, τ_Y), a mapping $f : X \to Y$ is called continuous if $V \in \tau_Y \Longrightarrow f^{-1}(V) \in \tau_X$.*

The first time when both sets and the topological structure are allowed to be fuzzy as well was considered in [20],[33] (independently) and further extended to the most general case, when the codomains of fuzzy sets and the codomains of fuzzy topologies may also differ, in the following definition:

Definition 5. [21], [22], [36]. *Let L, M be complete infinitely distributive lattices with $\mathbf{0}^L, \mathbf{0}^M, \mathbf{1}^L, \mathbf{1}^M$ their bottom and top elements respectively. A mapping $\mathcal{T} : L^X \to M$ is called an (L, M)-fuzzy topology on a set X if the following conditions are satisfied: (1) $\mathcal{T}(\mathbf{0}_X^L) = \mathcal{T}(\mathbf{1}_X^L) = \mathbf{1}^M$; (2) $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ $\forall A, B \in L^X$; (3) $\mathcal{T}(\bigvee_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{T}(A_i)$ $\forall \{A_i \mid i \in I\} \subseteq L^X$. The corresponding pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space. Given two (L, M)-fuzzy topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a mapping $f : X \to Y$ is called continuous if $\mathcal{T}_Y(V) \leq \mathcal{T}_X(f^{-1}(V))$ for every $V \in L^Y$.*

Fuzzy co-topologies In "classic topology" closed sets are defined as complements of open sets thus determining a bijection between families of open and closed sets. A similar, definite, situation appears also in the case of $[0, 1]$-topologies (i.e. Chang fuzzy topologies): the complementation $[0, 1]$ defined by subtraction naturally provides bijection between a family of fuzzy sets and the family of comlements of these fuzzy sets. On the other hand, in the case of arbitrary L-topologies, the absence of the naturally defined complementation operation on the lattice L leads to the need for independent development of theories of (different versions) of fuzzy topologies and the theories of (the corresponding versions) of fuzzy co-topologies. For the first time, as far as we know, this approach was announced in [4] and was subsequently developed in the articles by M.L. Brown and his co-authors [5], [6], et al. Below we recall basic concepts related to fuzzy co-topologies.

Definition 6. [7],[26],[8]. *A family of L-fuzzy subsets $\sigma \subseteq L^X$ is called an L-co-topology on a set X if the following conditions are satisfied: (1) $\mathbf{0}_X, \mathbf{1}_X \in \sigma$; (2) $A, B \in \sigma \implies A \vee B \in \sigma$; (3) $\{A_i \mid i \in I\} \subseteq \sigma \implies \bigwedge_{i\in I} A_i \in \sigma$. The corresponding pair (X, σ) is called an L-co-topological space.*

Definition 7. [21], [22], [36]. *A mapping $\mathcal{S} : L^X \to M$ is called an (L, M)-fuzzy co-topology on a set X if the following conditions are satisfied: (1) $\mathcal{S}(\mathbf{0}_X^L) = \mathcal{S}(\mathbf{1}_X^L) = \mathbf{1}^M$; (2) $\mathcal{S}(\mathbf{0}_X^L) = \mathcal{S}(\mathbf{1}_X^L) = \mathbf{1}^M$; (3) $\mathcal{S}(\bigwedge_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{S}(A_i)$ $\forall \{A_i \mid i \in I\} \subseteq L^X$. The corresponding pair $(X, \mathcal{S})$ is called an (L, M)-fuzzy co-topological space.*

3 Bipolar fuzzy extensions of *L*-topologies and *L*-co-topologies

Openness degrees of fuzzy subsets in *L*-topological spaces Let (X, τ) be an L-topological space. Given an L-fuzzy set $A \in L^X$, we define its interior A^o by setting $A^o = \bigvee\{U \mid U \in \tau, U \leq A\}$. It is well known (see, e.g. [31]) and easy to see that A^o is the largest open fuzzy set contained in A.

We use the interior A^o of A and relation of inclusion $\hookrightarrow$ in order to measure *the degree of openness of an L-fuzzy set A in an L-topological space (X, τ)* by setting $\mathcal{T}_\tau^+(A) = A \hookrightarrow A^o$. By varying fuzzy sets A over L^X, we get a mapping $\mathcal{T}_\tau^+ : L^X \to L$.

Applying the properties of relation $\hookrightarrow$: $L^X \to L^X \to L$, we can prove the following result.

Theorem 2. *The mapping* $\mathcal{T}_\tau^+ : L^X \to L$ *is an* (L, L)*-fuzzy topology on the set* X*. If a function* $f : (X, \tau_X) \to (Y, \tau_Y)$ *of* L*-topological spaces is continuous, then the mapping* $f : (X, \mathcal{T}^+_{\tau_X}) \to (Y, \mathcal{T}^+_{\tau_Y}))$ *of the corresponding (L,L)-fuzzy topological spaces is continuous, too.*

Non-openness degrees of *L*-fuzzy subsets in *L*-topological spaces Let (X, τ) be an L-topological space, let A be its L-fuzzy subset, and let $A^o = \bigvee\{U \mid U \in \tau, U \leq A\}$ be its interior.

Definition 8. *The degree of non-openness of a fuzzy set* A *in an* L*-topological space is defined by* $\mathcal{T}_\tau^-(A) =^\sim (A \not\leftrightarrow A^o)$*. By varying* A *over* L^X *we get an operator* $\mathcal{T}_\tau^- : L^X \to L^-$ *of non-openness in the* L*-topological space* (X, τ)*.*

Applying the properties of relation $\not\leftrightarrow : L^X \times L^X \to L$ established in Proposition 4, we can prove the following result.

Theorem 3. *The mapping* $\mathcal{T}_\tau^- : L^X \to L^-$ *is an* (L, L^-)*-fuzzy topology on the set* X*.*

Bipolar $\mathcal{L}$-fuzzy topology on an *L*-topological space In the previous paragraphs we have defined two fuzzy topologies on a topological space X: an (L, L)-fuzzy topology $\mathcal{T}^+ : L^X \to L$ that determines the openness degree of a fuzzy set $A \in L^X$ and an (L, L^-)-fuzzy topology $\mathcal{T}^- : L^X \to L^-$ that determines its non-openness degree. Basing on these fuzzy topologies we define a mapping $\mathcal{T}_\tau : L^X \to \mathcal{L}$ by setting:

$$\mathcal{T}_\tau(A) = (\mathcal{T}_\tau^+(A), \mathcal{T}_\tau^-(A)) \in \mathcal{L} \text{ for every } A \in L^X.$$

On the base of theorems 2 and 3 we easily get the following main result of this section:

Theorem 4. *The mapping* $\mathcal{T} : L^X \to \mathcal{L}$ *is an* $(L, \mathcal{L})$*-fuzzy topology on* X*.*

Since the lattice $\mathcal{L}$ is a bipolarization of the original lattice L and wishing to emphasize the role of bipolarity in our constructions, we will refer to the $(L, \mathcal{L})$-fuzzy topology $\mathcal{T} : L^X \to \mathcal{L}$ also as the bipolar $\mathcal{L}$-fuzzy topology induced by the L-topology $\tau \subseteq L^X$

Remark 1. In [27], see also [28], a stronger version of an (Chang-Goguen) L-topological space was introduces by replacing axiom (1) with a stronger axiom requesting that all constant fuzzy sets α_X must be included in τ. Following [31] such L-topological spaces are called *stratified.* We can easily get the following stratified version of Theorem 4:
If τ *is a stratified* L*-topology on a set* X *then* $\mathcal{T} : L^X \to \mathcal{L}$ *is a stratified bipolar* $\mathcal{L}$*-fuzzy topology on the set* X *that is a bipolar* $\mathcal{L}$*-fuzzy topology such that*
(1^s) $\mathcal{T}(a_X) = (\mathcal{T}^+(a_X), \mathcal{T}^-(a_X)) = (\mathbf{1}, \mathbf{0}^-) = \top_{\mathcal{L}}$ *for every* $a \in L$*.*

3.1 Bipolar fuzzy extension of L-co-topologies

Closedness degrees of fuzzy subsets in L-co-topologcal spaces Let (X,σ) be an L-co-topological space. Given an L-fuzzy set $A \in L^X$, we define the closure $\bar{A}$ of A in the space (X,σ), by setting $\bar{A} = \bigwedge\{F \mid F \in \sigma, F \geq A\}$.

We use the closure $\bar{A}$ of A to measure its *degree of closedness* $\mathcal{S}^+_\sigma(A)$ in the space (X,σ) by setting $\mathcal{S}^+_\sigma(A) = \bar{A} \hookrightarrow A$. By varying fuzzy sets A over L^X, we get a mapping $\mathcal{S}^+_\sigma : L^X \to L$. Now, applying the properties of relation $\hookrightarrow : L^X \times L^X \to L$, we can prove the following result.

Theorem 5. *The mapping $\mathcal{S}^+_\sigma : L^X \to L$ is an (L,L)-fuzzy co-topology on the set X. If $f : (X,\sigma_X) \to (Y,\sigma_Y)$ is a continuous mapping of L-co-topological spaces, then $f : (X,\mathcal{S}^+_{\sigma_X}) \to (Y,\mathcal{S}^+_{\sigma_Y})$ is a continuous mapping of the corresponding (L,L)-fuzzy co-topological spaces.*

Non-closedness degrees of fuzzy subsets in L-co-topological spaces Patterned after the definition of non-openness degree for L-fuzzy subsets of an L-topological space, we introduce here the degree of non-closedness for L-fuzzy subsets $A \in L^X$ in an L-co-topological space (X,σ) by setting $\mathcal{S}^-_\sigma(A) =^\sim \bar{A} \not\hookrightarrow A$. Varying A over L^X, we get an operator $\mathcal{S}^-_\sigma : L^X \to L^-$ of non-closedness of L-fuzzy subsets of the space (X,σ).

Referring to Proposition 4, we can establish the following result.

Theorem 6. *The mapping $\mathcal{S}^-_\sigma : L^X \to L^-$ is an (L,L^-)-fuzzy co-topology on the set X.*

Bipolar fuzzy co-topology on a co-topological space Basing on the fuzzy co-topologies $\mathcal{S}^+_\sigma : L^X \to L$ and $\mathcal{S}^-_\sigma : L^X \to L^-$ and applying theorems 5 and 6 we get the following main result of this section.

Theorem 7. *Mapping $\mathcal{S}_\sigma : L^X \to \mathcal{L}$ defined by*

$$\mathcal{S}_\sigma(A) = (\mathcal{S}^+_\sigma(A), \mathcal{S}^-_\sigma(A)) \in \mathcal{L} \text{ for every } A \in L^X$$

is an $(L,\mathcal{L})$-fuzzy co-topology on the set X.

4 Bipolar $\mathcal{L}$-fuzzy extension of an L-topology in case of a Girard monoid

Girard monoids as the framework for our studies The definitions of L-topologies and their internal properties depend only on the lattice structure of L. On the other hand, the model of extending of an L-topology to a bipolar $\mathcal{L}$-fuzzy topology is based on the conjunction $* : L \times L \to L$, the corresponding residuum $\mapsto : L \times L \to L$ and the unary operator ${}^c : L \to L$. Therefore the properties of the extended $\mathcal{L}$-fuzzy topologies depend on the interrelations between the conjunction $* : L \times L \to L$ and on complementation ${}^c : L \to L$. In this section we consider the special, in our opinion the most transparent case, when $(L, \leq, \wedge, \vee, *, \mapsto, {}^c)$ is a complete Girard monoid, see [10], [18].

Definition 9. [10] *A residuated lattice* $(L, \leq, \wedge, \vee, *, \mapsto)$ *is called a Girard monoid if* $(a \mapsto \mathbf{0}) \mapsto \mathbf{0} = a$ *for every* $a \in L$.

For every $a \in L$ in a Girard monoid we define $a^c = a \mapsto \mathbf{0}$. Thus we have connected in this special way all elements of our starting object, that is of the complete lattice $(L, \leq, \wedge, \vee, *, \mapsto, ^c)$: the conjunction $* : L \times L \to L$, the implication $\mapsto: L \times L \to L$ and the complementation $^c : L \to L$.

An important example of a Girard monoid is so called Łukasiewicz algebra, that is the tuple $([0,1], \leq, \wedge, \vee, *, \mapsto, ^c)$ where $a * b = \max\{a+b-1, 0\}$, $a \mapsto b = \min\{1-a+b, 1\}$ is the corresponding residuum, and $a^c = (a \mapsto 0) \mapsto 0 = 1-a$ for any $a, b \in [0,1]$

Remark 2. Note that a Girard monoid is a generalization of a well known concept of an MV-algebra that can be defined as a residuated lattice $(L, \leq, \wedge, \vee, *, \mapsto)$ satisfying $(a \mapsto b) \mapsto b = a \vee b$ for every $a, b \in L$, see, e.g. [14], [15].

Proposition 5. [18] *If* $(L, \leq, \wedge, \vee, *, \mapsto, ^c)$ *is a Girard monoid, then*

$$a \mapsto b = (a * b^c)^c = b^c \mapsto a^c \text{ for any } a, b \in L.$$

Relations between fuzzy topologies $\mathcal{T}_\tau^+$ and $\mathcal{T}_\tau^-$ in case of a Girard monoid. Directly from theorems 2 and 3 and the definition of a Girard monoid, we get the following theorem.

Theorem 8. *If* $(L, \leq, \wedge, \vee, *, \mapsto, ^c)$ *is a Girard monoid, then* $\mathcal{T}_\tau^+ = (^\sim\mathcal{T}_\tau^-)^c$, *that is* $\mathcal{T}_\tau^+(A) = (^\sim\mathcal{T}_\tau^-(A))^c$ *for every* $A \in L^X$.

Corollary 1. *The bipolar $\mathcal{L}$-fuzzy topology $\mathcal{T}_\tau$ induced by an L-topology τ on a set X in case of a Girard monoid $(L, \leq, \wedge, \vee, \mapsto)$ is defined by*

$$\mathcal{T}_\tau(A) = (\mathcal{T}_\tau^+(A), ^\sim \mathcal{T}_\tau^+(A)) \text{ for every } A \in L^X.$$

In particular, if $(L, \leq, \wedge, \vee, \mapsto)$ is the Łukasiewicz algebra, then
$\mathcal{T}_\tau(A) = (\mathcal{T}_\tau^+(A), 1 - \mathcal{T}_\tau^+(A))$.

Specifically, if $L = \{0, \frac{1}{2}, 1\}$ is the Łukasiewicz algebra, we have the tautology : *a fuzzy set is open if and only if it is not non open* and *a fuzzy set is half open if and only if it is half non open*

Relations between fuzzy co-topologies $\mathcal{S}+$ and $\mathcal{S}^-$ in case of a Girard monoid Results similar to the ones stated in the previous subsection for bipolar $\mathcal{L}$-fuzzy topology $\mathcal{T}_\tau$ induced by an L-topology τ are valid also for the $\mathcal{L}$-fuzzy co-topology $\mathcal{S}_\sigma$ induced by an L-co-topology σ.

Theorem 9. *If* $(L, \leq, \wedge, \vee, *, \mapsto, ^c)$ *is a Girard monoid and* (X, σ) *is an L-co-topological space, then* $\mathcal{S}_\sigma^+ = (^\sim\mathcal{S}_\sigma^-)^c$, *i.e.* $\mathcal{S}_\sigma^+(A) = (^\sim\mathcal{S}_\sigma^-(A))^c$ *for every* $A \in L^X$.

Corollary 2. *The bipolar $\mathcal{L}$-fuzzy co-topology induced by an L-co-topology σ on a set X in case of a Girard monoid $(L, \leq, \wedge, \vee, *, \mapsto)$ is defined by* $\mathcal{S}_\sigma(A) = (\mathcal{S}_\sigma^+(A), ^\sim \mathcal{S}_\sigma^+(A))$ *for every* $A \in L^X$. *In particular, in case* $(L, \leq, \wedge, \vee, *, \mapsto)$ *is a Łukasiewicz algebra,* $\mathcal{S}_\sigma(A) = (\mathcal{S}_\sigma^+(A), 1 - \mathcal{S}_\sigma^+(A))$.

Relations between bipolar $\mathcal{L}$-fuzzy topology and bipolar $\mathcal{L}$-fuzzy co-topology in case of a Girard monoid Turning to the relationships between the bipolar extensions of L-topologies and L-co-topologies in case when L is a Girard monoid, notice first that in this case it is possible, (thanks to double negation law $(a^c)^c = a \ \forall a \in L$) to consider an L-co-topology as dual to the corresponding L-topology. Namely, given an L-topology $\tau \subseteq L^X$ on a set X the corresponding L-co-topology $\tau^c =_{def} \sigma$ is defined as $\sigma = \{A \in L^X \mid A^c \in \tau\}$. Hence in case of a Girard monoid, when speaking about an L-topology, we also mean the corresponding L-co-topology This correspondence between L-topology and L-co-topology allows us to establish the following known (see, e.g. [26]) and easy provable connection between closure and interior operators in an L-topological space.

Proposition 6. *Given an L-topological space (X,τ) and an L-fuzzy set $A \in L^X$, the following relationships hold between the interior and the closure operators:* $\bar{A} = ((A^c)^o)^c$, $(\bar{A})^c = (A^c)^o$, $A^o = (\overline{A^c})^c$ $(A^o)^c = \overline{A^c}$.

Referring to this statement, we can establish connections between bipolar $\mathcal{L}$-fuzzy topology and bipolar $\mathcal{L}$-fuzzy co-topology induced by an L-topology τ on X.

Theorem 10. *Let (X,τ) be an L-topological space where $(L,\leq,\wedge,\vee,*,\mapsto,^c)$ is a Girard monoid. Then $\mathcal{T}^+_\tau(A) = \mathcal{S}^+_\sigma(A^c)$ for every $A \in L^X$.*

Theorem 11. *Let (X,τ) be an L-topological space, where $(L,\leq,\wedge,\vee,*,\mapsto,^c)$ is a Girard monoid. Then $\mathcal{T}^-_\tau(A) = \mathcal{S}^-_\sigma(A^c)$ for every $A \in L^X$.*

Corollary 3. *Let (X,τ) be an L-topological space, where $(L,\leq,\wedge,\vee,*,\mapsto,^c)$ is a Girard monoid. Then the bipolar $\mathcal{L}$-fuzzy topology $\mathcal{T}_\tau : L^X \to \mathcal{L}$ and the bipolar $\mathcal{L}$-fuzzy co-topology $\mathcal{S}_\sigma : L^X \to L$ generated by τ are respectively:*

$$\mathcal{T}_\tau(A) = \left(\mathcal{T}^+_\tau(A), \mathcal{T}^-_\tau(A)\right) \text{ and } \mathcal{S}_\sigma(A) = \left(\mathcal{T}^+_\tau(A^c), \mathcal{T}^-_\tau(A^c)\right) \text{ for every } A \in L^X.$$

4.1 Extension of L-topologies to bipolar $\mathcal{L}$-fuzzy topologies viewed in the framework of category theory

In order to obtain an accurate mathematical justification and a clear perspective for the further development of the bipolar fuzzy extension model presented here, it is necessary to consider this model also from the point of view of category theory. We are currently working on this issue. Some partial results obtained in this direction in the case when L is a Girard monoid are presented in the following theorem, the proof of which, together with other results in this direction, will be presented in the extended version of the article.

Theorem 12. *Assigning to an L-topological space (X,τ) the $(L,\mathcal{L})$-fuzzy topological space $\Phi(X,\tau) = (X,(\mathcal{T}^+_\tau,\mathcal{T}^-_\tau)$ and viewing a continuous mapping $f : (X,\tau_X) \to (Y,\tau_Y)$ as a mapping $\Phi(f) : (X,\left(\mathcal{T}^+_{\tau_X},\mathcal{T}^-_{\tau_X}\right)) \to (Y,\left(\mathcal{T}^+_{\tau_Y},\mathcal{T}^-_{\tau_Y}\right))$,* *we obtain an embedding functor $\Phi : L$-**TOP** $\to \mathcal{L}$-**FTOP** of the category L-**TOP** of L-topological spaces into the category $(L,\mathcal{L})$-**FTOP** of $(L,\mathcal{L})$-fuzzy topological spaces. On the other hand, by assigning to an $(L,\mathcal{L})$-fuzzy topological space $(X,(\mathcal{T}^+_X,\mathcal{T}^-_X))$ the L-topological space $\Psi(X,(\mathcal{T}^+_X,\mathcal{T}^-_X)) = (X,\tau_{\mathcal{T}^+_X})$ where $\tau_{\mathcal{T}^+} = \{A \in L^X : \mathcal{T}^+(A) = \mathbf{1}\}$ and interpreting a continuous mapping $f : (X,\big(\mathcal{T}^+_X,\mathcal{T}^-_X)\big) \to \big(Y,(\mathcal{T}^+_Y,\mathcal{T}^-_Y)\big)$ as the mapping $\Psi(f) : (X,\tau_{\mathcal{T}^+_X}) \to (Y,\tau_{\mathcal{T}^+_Y})$, we obtain a functor $\Psi : (L,\mathcal{L})$-**FTOP** $\to L$-**TOP**. This functor is right inverse of the embedding functor $\Phi : L$-**TOP** $\to (L,\mathcal{L})$-**FTOP** i.e. $\Psi\circ\Phi : L$-**TOP** $\to L$-**TOP** is the identity functor.*

5 Conclusion

We consider this work as the first example of the use of bipolar lattices in studying properties of fuzzy topological spaces. Implementing this idea, we presented a model for fuzzification of L-topological spaces. The essence of this model is the transition from the case of a structure, when only sets are fuzzy (in our case this is an L-topology), to the case when the structure itself becomes fuzzy with a bipolar scale of values (in our case this is the $(L,\mathcal{L})$-fuzzy topology). As a by-product of this model, we consider the possibility of a deeper qualitative analysis of topologies in the context of fuzzy sets and their specific properties.

In our work we distinguish two parts. The first, presented in Section 3, develops this model in the most general context, that is, in a situation where no relationships are assumed between the operators $*$ and ^{c} on the residuated lattice L. No specific relationship between the degrees of openness and the degrees of non-openness of fuzzy sets can be expected in this case. Therefore, in this case, we use general bipolar scales to estimate these degrees. On the other hand, in Section 4 we limit the scope of our study to the case when the resulting lattice L is a Girard monoid. In this case the role of the bipolarity of the lattice in evaluation becomes especially transparent and the relationships between the degrees of the presence of the property (in our case openness or closedness) and the presence of the opposite property (in our case, non-openness and non-closedness) become coordinated. Moreover, in this case our bipolar $(L,\mathcal{L})$-fuzzy topology can be interpreted as intuitionistic L-fuzzy topology (c.f., e.g [29]), that is as a pair of mappings $\mathcal{T}^+ : L^X \to L, T^- : L^X \to L$ describing the degrees of openess and non-openness of L-fuzzy sets. (See, for example [24] or [39] for a general discussion about the relationships and conceptional differences between approaches based on bipolar lattices and "intuitionistic"-based approaches to the assessment of the degrees of presence vs non-presence of properties vs opposite properties.)

Limitation of the volume of the article did not allow us to provide proofs for the presented results. We plan to include them, along with other results related to this topic, in a further revised version of this paper. In paticular, categorical aspects of this model will be worked out and presented there in detail.

As for the prospects for continuing the work begun in this article, as the first task we consider the use of a bipolarization model to study specific topological properties. As stated in the introduction, we assume that the use of bipolar lattices to analyze such basic topological properties as compactness, connectedness and separation can provide qualitatively new information about the structure (of different variants) of fuzzy topological spaces. Further, if the use of bipolar lattices proves useful within the framework of fuzzy topology, we can try to apply them to the bipolar extension of other fuzzy mathematical systems, in particular fuzzy algebraic structures.

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