On Pseudo-Homogeneity of Fuzzy Implications

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Abstract. The study of functional equations involving fuzzy logic connectives, especially fuzzy implications, has found immense utility both in the advancement of theory and applications. In this work, we discuss the pseudo-homogeneity functional equation involving fuzzy implications. An interesting outcome of this work is a novel sufficient condition on the triple $(S, T_{\mathbf{P}}, N)$ such that the QL-operation obtained from it is also a QL-implication.

Keywords: Pseudo-Homogeneity · Fuzzy Implications · QL Implications.

1 The Pseudo-Homogeneity Functional Equation

The study of functional equations involving fuzzy logic connectives (FLCs) has long been a topic of importance and utility. Depending on the class of FLCs considered, one has a plethora of such equations to study from.

One such functional equation that has been studied for the class of associative and/or commutative FLCs is that of Homogeneity, viz.,

$$F(\lambda \cdot x, \lambda \cdot y) = \lambda \cdot F(x, y) \ , \qquad x, y, \lambda \in [0, 1] \ , \tag{Hom}$$

and its various generalisations, viz., pseudo- or quasi-homogeneity, see for instance the works of [2,6] dealing with t-norms and t-conorms, and [5] on overlap functions. While many of these works are largely theoretical in nature, the work of Lima et al. [4] dealing with the pseudo-homogeneity of t-subnorms does offer potential applications in multi-expert decision-making problems.

A study of the homogeneity functional equation involving fuzzy implications has not been done. This is because the properties of a fuzzy implication force λ to be equal to 1 and hence (Hom) is no more interesting.

However, fuzzy implications are amenable towards a generalised form of homogeneity, termed pseudo-homogeneity in the literature, defined as given below.

Definition 1. *Let I be a fuzzy implication, and let $F : [0, 1]^2 \to [0, 1]$. Then I is said to satisfy pseudo-homogeneity with respect to F, if*

$$I(\lambda \cdot x, \lambda \cdot y) = F(\lambda, I(x, y)), \tag{PH}$$

for any $\lambda, x, y \in [0, 1]$.

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1.1 A Quick Outline

In this work, we study the above functional equation (PH). Following a brief section presenting the required preliminaries, in Section 3, we present our nascent results on the pseudo-homogeneity functional equation involving fuzzy implications. In Section 4, we characterise fuzzy implications satisfying the neutrality property that are self pseudo-homogeneous, and in Section 5, we present some sufficient conditions under which a fuzzy implication is pseudo-homogeneous with respect to another fuzzy implication. An interesting outcome of this study is a novel sufficient condition on the triple $(T_{\mathbf{P}}, S, N)$, where $T_{\mathbf{P}}$ is the product t-norm, S is any t-conorm, and N is a continuous negation, under which $I_{(T_{\mathbf{P}},S,N)}$ is a fuzzy implication. Finally, we present some concluding remarks.

2 Preliminaries

In this section, we define fuzzy negation and fuzzy implication and take a look at some of their examples. We also define certain families of fuzzy implications. For definitions of t-norm and t-conorm, we refer the readers to [3].

Definition 2. *A function $N : [0,1] \to [0,1]$ is said to be a fuzzy negation if for any $x_1, x_2 \in [0,1]$,*

(i) $x_1 \leq x_2 \implies N(x_2) \geq N(x_1)$, i.e., N is decreasing.
(ii) $N(0) = 1$, and $N(1) = 0$.

Some fuzzy negations are listed in the following example.

Example 1. (i) $N_C(x) = 1 - x$ is a continuous fuzzy negation.
(ii) $N_1(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{else.} \end{cases}$ is the largest fuzzy negation.
(iii) $N_0(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{else.} \end{cases}$ is the least fuzzy negation.

Definition 3. *[1] A function $I : [0,1]^2 \to [0,1]$ is said to be a* ***fuzzy implication*** *if the following properties hold for any $x_1, x_2, y_1, y_2, x, y \in \mathcal{X}$:*

(i) $x_1 \leq x_2 \implies I(x_2, y) \leq I(x_1, y)$, i.e., $I(\cdot, y)$ is decreasing.
(ii) $y_1 \leq y_2 \implies I(x, y_1) \leq I(x, y_2)$, i.e., $I(x, \cdot)$ is increasing.
(iii) $I(0,0) = 1$, $I(1,1) = 1$, and $I(1,0) = 0$.

We shall denote the set of all fuzzy implications by $\mathbb{I}$.

A few basic examples of fuzzy implications can be seen in Table 1.

Remark 1. Note that given a fuzzy implication I, $N_I : [0,1] \to [0,1]$ defined as $N_I(x) = I(x, 0)$ is a fuzzy negation.

Table 1. Some examples of fuzzy implications

Name	**Formula**
Lukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \ , \\ y, & \text{else} \ . \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \ , \\ 0, & \text{else} \ . \end{cases}$
$I_{\mathbf{1}}$	$I_{\mathbf{1}}(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,0) \ , \\ 1, & \text{else} \ . \end{cases}$
$I_{\mathbf{0}}$	$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \ , \\ 0, & \text{else} \ . \end{cases}$

Definition 4. *A fuzzy implication I is said to satisfy the neutrality property if*

$$I(1,x) = x, \qquad x \in [0,1]. \tag{NP}$$

In literature, various classes of fuzzy implications have been defined. Two classes, (S,N)- and QL implications, are defined below.

(S,N)-implications are a generalisation of the material implication of classical logic to the setting of fuzzy logic and are defined as follows.

Definition 5. *An $I \in \mathbb{I}$ is called an **(S,N)-implication**, denoted $I_{S,N}$, if there exist a t-conorm S, and a fuzzy negation N such that,*

$$I_{S,N}(x,y) = S(N(x),y) \ , \qquad x,y \in [0,1] \ .$$

QL-implications are obtained as a generalisation of the following implication employed in quantum logic, viz., $p \implies q = \neg p \vee (p \wedge q)$, to the setting of fuzzy logic.

Definition 6. *An $I \in \mathbb{I}$ is called a **QL-implication**, denoted $I_{T,S,N}$, if there exist a t-norm T, a t-conorm S, and a fuzzy negation N such that*

$$I_{T,S,N}(x,y) = S(N(x), T(x,y)) \ , \qquad x,y \in [0,1] \ .$$

3 Pseudo-Homogeneity: Some Necessary Conditions

We begin by showing certain examples of fuzzy implications that satisfy (PH).

Example 2. (i) The Lukasiewicz implication $I_{\mathbf{LK}}$ satisfies (PH) with Reichenbach implication $I_{\mathbf{RC}}$.

(ii) The Weber implication $I_{\mathbf{WB}}$, and $I_{\mathbf{1}}$ satisfy (PH) with themselves.

(iii) Consider the Gödel implication

$$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases} .$$

Then $I_{\mathbf{GD}}$ satisfies (PH) with

$$F_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } y = 1 \text{ or } x = 0, \\ x \cdot y, & \text{else.} \end{cases} .$$

We now discuss some necessary conditions on F for a fuzzy implication I to satisfy (PH) with it.

Lemma 1. *Let I be a fuzzy implication and $F : [0,1]^2 \to [0,1]$ be such that (I,F) satisfies* (PH)*. Then*

(i) $F(0,\alpha) = 1$, $\alpha \in \mathcal{R}an(I)$, *where* $\mathcal{R}an(I)$ *denotes the range of* I.
(ii) $F(\lambda,1) = 1$, $\lambda \in [0,1]$.

Proof. (i) For any $x,y \in [0,1]$, $1 = I(0,0) = I(0 \cdot x, 0 \cdot y) = F(0, I(x,y))$. Thus, $F(0,\alpha) = 1$ for all $\alpha \in \mathcal{R}an(I)$.
(ii) For any $\lambda \in [0,1]$, $F(\lambda,1) = F(\lambda, I(0,0)) = I(\lambda \cdot 0, \lambda \cdot 0) = I(0,0) = 1$.

Remark 2. (i) Notice that $F(0,\alpha)$ need not be 1 for all $\alpha \in [0,1]$. Consider for instance the Rescher implication $I_{\mathbf{RS}}$ and let

$$F(x,y) = \begin{cases} 1, & \text{if } y = 1 \text{ or } (x,y) \in (\{0\} \times [0,1] \setminus \frac{1}{2}) \cup (\{1\} \times \{\frac{1}{2}\}), \\ 0, & \text{if } (x,y) \in (0,1] \times \{0\}, \\ 1-y, & \text{else.} \end{cases}$$

It can be verified that $(I_{\mathbf{RS}}, F)$ satisfies pseudo-homogeneity and $F(0,\frac{1}{2}) = \frac{1}{2} \neq 1$.
(ii) Note that F in the example above is neither increasing nor decreasing in both the variables. Thus, F need not satisfy any monotonicity condition in any variable.
(iii) Note that $I_{\mathbf{RS}}$ satisfies pseudo-homogeneity with itself as well, i.e., it is self pseudo-homogeneous. Thus, for a pseudo-homogeneous I, there may not exist a unique F such that (I,F) satisfies (PH).

Theorem 1. *If a fuzzy implication I satisfies* (NP)*, and $F : [0,1]^2 \to [0,1]$ be such that (I,F) satisfies* (PH)*, then*

(i) F *is unique, and* $F(x,y) = I(x, x \cdot y)$.
(ii) F *is continuous if and only if* I *is continuous.*
(iii) F *is commutative in the open unit square* $(0,1)^2$ *if and only if for all* $x,y \in (0,1)$, $I(x, x \cdot y) = I(y, x \cdot y)$.

Proof. (i) $F(x,y) = F(x, I(1,y)) = I(x \cdot 1, x \cdot y) = I(x, x \cdot y)$.

(ii) From (i) it follows that if I is continuous, F is continuous. Suppose, F is continuous. Define $\mathcal{A} = \{(x, y) \,|\, x \leq y\}$, and $\mathcal{B} = \{(x, y) \,|\, x > y\}$. Then I is continuous on $\mathcal{A}$ since $I(x, y) = 1$ whenever $x \leq y$. Furthermore, $I(x, y) = F(x, \frac{y}{x})$ whenever $x > y$. Thus I is continuous on $\mathcal{B}$. Thus, to prove that I is continuous, we only need to check the continuity of I on the diagonal. Let $x \in [0, 1]$, and $(y_n)_{n\in\mathbb{N}}$ be a sequence in $[0, x)$ such that $y_n \to x$, where $\mathbb{N}$ is the set of natural numbers. Then

$$\lim_{y_n\to x} I(x, y_n) = \lim_{y_n\to x} F(x, \frac{y_n}{x}) = F(x, \lim_{y_n\to x} \frac{y_n}{x}) = F(x, 1) = 1 = I(x, x).$$

It follows that I is continuous on the diagonal.

(iii) For any $x, y \in (0, 1)$,

$$\begin{aligned} I(x, x \cdot y) = I(y, x \cdot y) &\iff I(x \cdot 1, x \cdot y) = I(y \cdot 1, y \cdot x) \\ &\iff I(x \cdot 1, x \cdot y) = I(y \cdot 1, y \cdot x) \\ &\iff F(x, I(1, y)) = F(y, I(1, x)) \\ &\iff F(x, y) = F(y, x) \end{aligned}$$

Example 3. Consider the Gödel implication $I_{\mathbf{GD}}$, mentioned in Example 2(iii). Then $I_{\mathbf{GD}}$ satisfies (PH) with $F_{\mathbf{GD}}$ which is commutative in $(0, 1)^2$. Clearly, $I_{\mathbf{GD}}(x, x \cdot y) = x \cdot y = I_{\mathbf{GD}}(y, y \cdot x)$, since $x \cdot y < \min(x, y)$ for all $x, y \in (0, 1)$.

In the following result, we take a look at a necessary condition on I, given (I, F) satisfies (PH) for some $F : [0, 1]^2 \to [0, 1]$.

Lemma 2. *Let I be a fuzzy implication and $F : [0, 1] \to [0, 1]$ be such that (I, F) satisfies* (PH)*. Then $x \leq y$ implies $I(x, y) = 1$ for all $x, y \in [0, 1]$.*

Proof. Let $x, y \in [0, 1]$ be such that $x \leq y$. Then

$$I(x, y) = I(y \cdot \frac{x}{y}, y \cdot 1) = F(y, I(\frac{x}{y}, 1)) = F(y, 1) = 1.$$

Remark 3. Note that the converse of Lemma 2 may not be true. Consider for instance the Fodor implication

$$I_{\mathbf{FD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \max(1 - x, y), & \text{if } x > y. \end{cases}$$

$I_{\mathbf{FD}}$ is not pseudo-homogeneous with respect to any $F : [0, 1]^2 \to [0, 1]$ since if there did exist such an F, then from Theorem 1(i) we have $F(x, y) = I_{\mathbf{FD}}(x, xy) = \max(1 - x, xy)$. Consequently,

$$0.8 = I_{\mathbf{FD}}(0.2, 0.15) = I_{\mathbf{FD}}(\frac{1}{2}\cdot 0.4, \frac{1}{2}\cdot 0.3) = F(\frac{1}{2}, I_{\mathbf{FD}}(0.4, 0.3)) = F(\frac{1}{2}, 0.6) = \frac{1}{2},$$

which leads to a contradiction.

4 Self Pseudo-Homogeneity of Fuzzy Implications

In Example 2(ii), Weber implication and $I_{\mathbf{1}}$ are pseudo-homogeneous with respect to themselves. We shall call such implications self pseudo-homogeneous implications. These examples lead to the following question:

- Under what conditions is a fuzzy implication pseudo-homogeneous with respect to itself?

In this section, we answer this question. In the following result, we characterise self pseudo-homogeneous fuzzy implications satisfying the neutrality property.

Theorem 2. *If a fuzzy implication I satisfies* (NP)*, then the following are equivalent.*

(i) I is self pseudo-homogeneous.
(ii) $I(x,y) = \begin{cases} I_{\mathbf{WB}}(x,y), & \text{if } y > 0, \\ N(x), & \text{else.} \end{cases}$*, where $N = N_1$ or N_0 .*

Proof. It can be easily verified that (ii) implies (i). To prove the other way around, we begin by assuming that I is self-pseudo-homogeneous. By Lemma 2, we know that whenever $x \leq y$, $I(x,y) = 1$. By Theorem 1(i), we have $I(x,y) = I(x,xy)$. Now,

$$x > y \implies I(x,y) = I(x \cdot 1, x \cdot \frac{y}{x}) = I(x, I(1, \frac{y}{x})) = I(x, \frac{y}{x}).$$

Thus, if $x > y$ and $x < \frac{y}{x}$, we have $I(x, \frac{y}{x}) = I(x,y) = 1$. Also,

$$I(x,y) = I(x,xy) = I(x,x^2y) = ... = I(x,x^ny) = ...$$

which implies $I(x,z) = 1$ for all $0 < z < x$. Thus, to complete the proof, we only need to check how $I(x,y)$ behaves when $y = 0$.

Since I is a fuzzy implication, $I(x,0) = N(x)$ where $N : [0,1] \to [0,1]$ is a fuzzy negation. Now,

$$N(\lambda \cdot x) = I(\lambda \cdot x, 0) = I(\lambda \cdot x, \lambda \cdot 0) = I(\lambda, I(x,0)) = I(\lambda, N(x)).$$

If $N(x) > 0$, by the nature of I, we get $N(\lambda \cdot x) = 1$ for all $\lambda < 1$. Thus, if $N(x) > 0$, $N(y) = 1$ for all $y < x$.
Suppose $1 > N(x_0) = a > 0$. Then $N(y) = 0$ for $y > x_0$ since if $N(y) > 0$, that would imply $N(x_0) = 1$, which is a contradiction. Now, there exist $\lambda, x \in (0,1)$ such that $\lambda, x > x_0$ and $\lambda \cdot x = x_0$. Thus,

$$N(x_0) = N(\lambda \cdot x) = I(\lambda, N(x)) = I(\lambda, 0) = N(\lambda) = 0.$$

This leads to a contradiction since $N(x_0) = a > 0$. Thus N only takes values from $\{0,1\}$.

Let $t_0 = \sup\{t \in [0,1] \mid N(t) = 1\}$. Suppose $0 < t_0 < 1$. Let $t', t* \in [0,1]$ such that

$$t_0 t' < t* < t_0 < t' < 1.$$

Thus $N(t') = 0$. Furthermore, $t* = \frac{t*}{t'} \cdot t' = \lambda \cdot t'$, where $\lambda = \frac{t*}{t'} > t_0$. Thus,

$$0 = N(\lambda) = I(\lambda, 0) = I(\lambda, N(t')) = N(\lambda \cdot t') = N(t*) = 1,$$

which is absurd. Hence, $t_0 = 1$ or $t_0 = 0$.

Thus, $I(x,0) = N_1(x)$ or $I(x,0) = N_0(x)$. Thus, $I(x,y) = \begin{cases} I_{\mathbf{WB}}(x,y), & \text{if } y > 0, \\ N(x), & \text{else.} \end{cases}$, where $N = N_0$ or N_1.

5 Fuzzy Implications pseudo-homogeneous with respect to other fuzzy implications

In Example 2(i), the Lukasiewicz implication is pseudo-homogeneous with respect to another fuzzy implication, namely the Reichenbach implication. The example leads us to the following question:

- Under what conditions is a fuzzy implication pseudo-homogeneous with respect to another fuzzy implication?

In this section, we present a partial answer to this question. In the following result, we present certain sufficient conditions under which a fuzzy implication is pseudo-homogeneous with respect to another fuzzy implication.

Theorem 3. *Let I be a fuzzy implication satisfying* (NP) *such that N_I is continuous and $F : [0,1]^2 \to [0,1]$ be such that (I, F) satisfies* (PH). *Then F is a fuzzy implication.*

Proof. Since N_I is continuous, it is onto. Furthermore,

$$N_I(\lambda \cdot x) = I(\lambda \cdot x, \lambda \cdot 0) = F(\lambda, I(x,0)) = F(\lambda, N_I(x)).$$

Thus, for any $b \in [0,1]$, there exists $c \in [0,1]$ such that $N_I(c) = b$, and $F(a,b) = F(a, N_I(c)) = N_I(ac)$. Now,

$$\begin{aligned} a_1 \le a_2 &\implies a_1 \cdot c \le a_2 \cdot c \\ &\implies N_I(a_1 \cdot c) \ge N_I(a_2 \cdot c) \\ &\implies F(a_1, b) \ge F(a_2, b). \end{aligned}$$

Thus F is decreasing in the first variable. Now let $b_1, b_2, c_1, c_2 \in [0,1]$ such that $N_I(b_1) = c_1$, and $N_I(b_2) = c_2$.

$$\begin{aligned} b_1 \le b_2 &\implies c_1 \ge c_2 \\ &\implies a \cdot c_1 \ge a \cdot c_2 \\ &\implies N_I(a \cdot c_1) \le N_I(a \cdot c_2) \\ &\implies F(a, b_1) \le F(a_2, b). \end{aligned}$$

Thus F is increasing in the second variable.

By Lemma 1, we can infer that $F(1,1) = 1 = F(0,0)$, and $F(1,0) = F(1, I(1,0)) = I(1\cdot 1, 1\cdot 0) = I(1,0) = 0$. Thus, F is a fuzzy implication.

Remark 4. Note that the converse of the above theorem need not be true, i.e. if I is a fuzzy implication satisfying (NP) such that N_I is continuous, and $F : [0,1]^2 \to [0,1]$ defined as $F(x,y) = I(x, x\cdot y)$ is a fuzzy implication, then (I,F) need not satisfy (PH).

Consider for instance $I_{\mathbf{DP}}(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 1-x, & \text{if } y = 0, \\ 1, & \text{else.} \end{cases}$

It can be verified that $I_{\mathbf{DP}}(x,xy) = I_{\mathbf{DP}}(x,y)$. However, $I_{\mathbf{DP}}$ doesn't satisfy (PH) with itself as

$$0.7 = I_{\mathbf{DP}}(0.5\cdot 0.6, 0.5\cdot 0) \neq I_{\mathbf{DP}}(0.5, I_{\mathbf{DP}}(0.6,0)) = I_{\mathbf{DP}}(0.5,0.4) = 1.$$

Corollary 1. *Let S be a t-conorm, and N be a continuous fuzzy negation such that $I_{S,N}$ satisfies* (PH) *with some F. Then $I_{T_{\mathbf{P}},S,N}$ is a QL-implication.*

Remark 5. (i) The Fodor implication $I_{\mathbf{FD}}$ given in Remark 3, is an $(S_{\mathbf{nM}}, N_C)$ implication. However, since $I_{(T_{\mathbf{P}},S_{\mathbf{nM}},N_C)}$ is not a QL implication, from Corollary 1 we can say that $I_{\mathbf{FD}}$ is not a pseudo-homogeneous fuzzy implication.

(ii) Note that the converse of the above corollary need not be true. Consider again for instance $I_{\mathbf{DP}}$. While it is a QL implication, it is also the corresponding (S,N) implication. However, it does not satisfy (PH).

6 Concluding Remarks

In this work, we studied the pseudo-homogeneity functional equation for fuzzy implications. We offered necessary conditions under which an implication satisfies pseudo-homogeneity. Furthermore, we characterised fuzzy implications which are pseudo-homogeneous with respect to themselves or other fuzzy implications. Interestingly, we see that pseudo-homogeneity of an (S,N) implication, where N is continuous, is a sufficient condition on the triple $(S, T_{\mathbf{P}}, N)$ for the QL-operation obtained from it to be a QL-implication.

References

1. Baczyński, M., Jayaram, B.: Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231. Springer-Verlag, Berlin Heidelberg (2008)
2. Ebanks, B.R.: Quasi-homogeneous associative functions. International Journal of Mathematics and Mathematical Sciences **21**, 351–357 (1998)
3. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms, Trends in Logic, vol. 8. Kluwer Academic Publishers, Dordrecht (2000)
4. Lima, L., Rocha, M., Lima, A.d., Bedregal, B., Bustince, H.: On pseudo-homogeneity of t-subnorms. In: 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). pp. 1–6 (2019)

5. Qiao, J., Hu, B.: On homogeneous, quasi-homogeneous and pseudo-homogeneous overlap and grouping functions. Fuzzy Sets and Systems **357**, 58–90 (2018)
6. Xie, A., Su, Y., Liu, H.: On pseudo-homogeneous triangular norms, triangular conorms and proper uninorms. Fuzzy Sets and Systems **287**, 203–212 (2016), theme: Aggregation Operations