Behavioral Dynamic Portfolio Selection via Epsilon-Contaminations

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Abstract. We consider a dynamic portfolio selection problem in a finite horizon binomial market model, composed of a non-dividend-paying risky stock and a risk-free bond. We assume that the investor's behavior distinguishes between gains and losses, as in the classical cumulative prospect theory (CPT). This is achieved by considering preferences that are represented by a CPT-like functional, depending on an S-shaped utility function. At the same time, we model investor's beliefs on gains and losses through two different epsilon-contaminations of the "real-world" probability measure. We formulate the portfolio selection problem in terms of the final wealth and reduce it to an iterative search problem over the set of optimal solutions of a family of non-linear optimization problems.

Keywords: Behavioral investor · S-shaped utility function · Epsilon-contamination · Dynamic portfolio selection.

1 Introduction

The *expected utility theory (EUT)* due to [19] is the standard normative approach to represent agent's preferences, used for portfolio selection, both in the static and the dynamic case (see, e.g., [4, 7]). One of the main reasons behind the success of EUT in the context of dynamic portfolio selection can be found in its mathematical simplicity, as it allows to recur to dynamic programming.

Nevertheless, the underlying assumptions behind EUT are that an agent is uniformly risk averse and has complete and unambiguous beliefs on final wealth, the latter expressed by a reference probability $\mathbf{P}$ (called "real-world" probability measure in finance [14]). Such assumptions have been challenged in the last decades by a series of "paradoxes" obtained from real agents, whose preferences are inconsistent with EUT (see, e.g., [1, 8]).

A more descriptive model has been introduced in [12] and then expanded in [18] to form the Nobel prize winner *cumulative prospect theory (CPT)*. This theory allows to overcome the main fallacies of EUT by distinguishing gains and losses: this is achieved by recurring to an S-shaped utility function and replacing probabilities with distorted probabilities for gains and losses, respectively.

The behavioral nature of CPT motivated research in portfolio selection that was initially confined in the static case, and then extended to the dynamic case (see [11] and the more recent papers [3, 9]). Despite its greater realism, portfolio selection in CPT shows many mathematical difficulties, nonetheless the failure of the dynamic programming approach. At the same time, a limitation of CPT can be found in the particular ambiguity structure that is assumed, encoded in two different probability distortions of the reference probability $\mathbf{P}$.

In this paper we consider a behavioral dynamic portfolio selection problem in a binomial market model [6], composed of a non-dividend-paying risky stock and a risk-free bond. The completeness of the market allows to formulate the dynamic portfolio selection in terms of the final wealth, reachable with a fixed initial endowment, the latter assumed to be the difference between the agent's initial wealth and a reference wealth.

We introduce a CPT-like functional still relying on an S-shaped utility function, but we model agent's beliefs recurring to two *epsilon-contaminations* of $\mathbf{P}$ (see, e.g., [10, 20]), related to gains and losses. In the particular case the initial endowment and the final wealth are restricted to be strictly positive, the loss term vanishes and we get back to the dynamic portfolio selection problem under ambiguity studied in [2, 15].

The peculiarity of epsilon-contaminations rests in the fact that they can be considered as neighborhood models around $\mathbf{P}$, determined by a suitable pseudo-distance [13]. Moreover, epsilon-contaminations present computational advantages in portfolio selection: *(i)* their envelopes are completely monotone/completely alternating capacites; *(ii)* they are determined by a finite (and manageable) set of extreme points; *(iii)* they are parameterized by a single parameter, that simplifies sensitivity analysis.

Thanks to the use of epsilon-contaminations, we can reduce the optimization to an iterative search problem over the set of optimal solutions of a family of non-linear optimization problems, parameterized by the set of gain states in the final wealth and the gain level of the initial endowment. Next, we show the application of the suggested procedure to a paradigmatic example.

The paper is structured as follows. Section 2 recalls the classical binomial market model. Section 3 introduces our behavioral portfolio selection problem, while Section 4 presents a paradigmatic example. Finally, Section 5 draws our conclusions and future perspectives. Due to space limitations, proofs have been omitted and reserved for an extended version of the present paper.

2 The Classical Multi-Period Binomial Market Model

The multi-period binomial model [6] refers to a perfect (competitive and frictionless) market under no-arbitrage, where two basic securities are traded: a non-dividend-paying stock and a risk-free bond.

For a finite horizon $T \in \mathbb{N}$, we denote by S_t and B_t the prices of the stock and the bond, respectively, at time $t \in \{0, \ldots, T\}$. The stochastic process $\{S_0, \ldots, S_T\}$ and the deterministic process $\{B_0, \ldots, B_T\}$ are such that $S_0 = s > 0$, $B_0 = 1$, and for $t = 1, \ldots, T$, the returns are

$$\frac{S_t}{S_{t-1}} = \begin{cases} u, \text{ with probability } p \\ d, \text{ with probability } 1-p \end{cases} \quad \text{and} \quad \frac{B_t}{B_{t-1}} = (1+r),$$

where $u > d > 0$ are the "up" and "down" stock price coefficients, r is the risk-free interest rate over each period, satisfying $u > (1+r) > d$, and $p \in (0,1)$ is the probability of an "up" movement for the stock price. Thus, for $t = 1, \ldots, T$, we have that

$$S_t = S_0 \prod_{n=1}^{t} \frac{S_n}{S_{n-1}} \quad \text{and} \quad B_t = (1+r)^t,$$

assuring that both price processes are strictly positive, in compliance with the limited liability assumption for securities (see, e.g., [14]). Notice that the trajectories of $\{S_0, \ldots, S_T\}$ can be represented graphically on a recombining binomial tree.

All the processes we consider are defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbf{P})$, where $\Omega = \{1, \ldots, 2^T\}$, $\mathcal{F} = 2^\Omega$ with 2^Ω the power set of Ω, and $\mathcal{F}_t$ is the algebra generated by random variables $\{S_0, \ldots, S_t\}$, for $t = 0, \ldots, T$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. As usual $\mathbf{E}^{\mathbf{P}}$ denotes the expected value with respect to $\mathbf{P}$. Moreover, we identify every $a \in \mathbb{R}$ with $a\mathbf{1}_\Omega$, where $\mathbf{1}_A$ denotes the indicator of $A \in \mathcal{F}$.

Assuming that the returns $\frac{S_1}{S_0}, \ldots, \frac{S_T}{S_{T-1}}$ are i.i.d. random variables, the probability $\mathbf{P}$ is completely singled out by the parameter p, and $\{S_0, \ldots, S_T\}$ is a multiplicative binomial process since

$$\mathbf{P}(S_t = u^k d^{t-k} s) = \frac{t!}{k!(t-k)!} p^k (1-p)^{t-k},$$

where S_t ranges in $\mathcal{S}_t = \{u^k d^{t-k} s \,:\, k = 0, \ldots, t\}$.

Let $V_0 \in \mathbb{R}$ be an initial endowment. A *self-financing strategy* $\{\theta_0, \ldots, \theta_{T-1}\}$ is an adapted process such that θ_t is the (random) number of shares of stock to buy (if positive) or short-sell (if negative) at time t up to time $t+1$ [4], that determines an adapted wealth process $\{V_0, \ldots, V_T\}$, where, for $t = 0, \ldots, T-1$,

$$V_{t+1} = (1+r)V_t + \theta_t S_t \left(\frac{S_{t+1}}{S_t} - (1+r) \right). \tag{1}$$

In turn, $V_t - \theta_t S_t$ is the amount of money invested in the bond from time t up to time $t+1$.

This market model is said to be *complete*, i.e., there is a unique "risk-neutral" probability measure $\mathbf{Q}$ on $\mathcal{F}$, equivalent to $\mathbf{P}$, such that the discounted wealth process of any self-financing strategy is a martingale under $\mathbf{Q}$:

$$\frac{V_t}{(1+r)^t} = \mathbf{E}_t^{\mathbf{Q}} \left[\frac{V_T}{(1+r)^T} \right], \tag{2}$$

for $t = 0, \ldots, T$, where $\mathbf{E}^{\mathbf{Q}}_t[\cdot] = \mathbf{E}^{\mathbf{Q}}[\cdot|\mathcal{F}_t]$ and $\mathbf{E}^{\mathbf{Q}}_0 \equiv \mathbf{E}^{\mathbf{Q}}$. Completeness implies that every payoff $V_T \in \mathbb{R}^{\Omega}$ depending only on the stock price history can be replicated by a dynamic self-financing strategy $\{\theta_0, \ldots, \theta_{T-1}\}$ and its unique no-arbitrage price at time $t = 0$ is determined by equation (2), since

$$V_0 = \frac{\mathbf{E}^{\mathbf{Q}}[V_T]}{(1+r)^T}. \tag{3}$$

Notice that the process $\{S_0, \ldots, S_T\}$ is still a multiplicative binomial process under $\mathbf{Q}$, completely characterized by the parameter

$$q = \frac{(1+r) - d}{u - d} \in (0, 1). \tag{4}$$

Both $\mathbf{P}$ and $\mathbf{Q}$ can be explicitly defined by identifying every state $i \in \Omega$ with the path of the stock price evolution corresponding to the T-digit binary expansion of number $i - 1$, in which ones are interpreted as "up" movements and zeros as "down" movements. Denoting by $\kappa(i)$ the number of "up" movements and by $T - \kappa(i)$ the number of "down" movements, it holds that (we avoid braces to simplify writing)

$$\mathbf{P}(i) = p^{\kappa(i)}(1-p)^{T-\kappa(i)} \quad \text{and} \quad \mathbf{Q}(i) = q^{\kappa(i)}(1-q)^{T-\kappa(i)}, \tag{5}$$

showing that both $\mathbf{P}$ and $\mathbf{Q}$ are strictly positive on $\mathcal{F} \setminus \{\emptyset\}$.

3 Behavioral Dynamic Portfolio Selection

Given the real-world probability $\mathbf{P}$ defined on $\mathcal{F}$ as in Section 2 (which is completely singled out by p) and $\epsilon \in (0, 1)$, the corresponding *epsilon-contamination model* (see, e.g., [10]) is the class of probability measures on $\mathcal{F}$ defined as

$$\mathcal{P}_{p,\epsilon} = \{\mathbf{P}' = (1-\epsilon)\mathbf{P} + \epsilon\mathbf{P}'' \,:\, \mathbf{P}'' \text{ is a probability measure on } \mathcal{F}\}.$$

The set of extreme points of $\mathcal{P}_{p,\epsilon}$ is

$$\text{ext}\,(\mathcal{P}_{p,\epsilon}) = \{\mathbf{P}^i \,:\, i = 1, \ldots, 2^T\}, \tag{6}$$

where

$$\mathbf{P}^i(j) = \begin{cases} (1-\epsilon)\mathbf{P}(j) + \epsilon, & \text{if } i = j, \\ (1-\epsilon)\mathbf{P}(j), & \text{if } i \neq j. \end{cases} \tag{7}$$

We have that the lower envelope $\nu_{p,\epsilon} = \min \mathcal{P}_{p,\epsilon}$ is defined on $\mathcal{F}$ as

$$\nu_{p,\epsilon}(A) = \begin{cases} (1-\epsilon)\mathbf{P}(A), & \text{if } A \neq \Omega, \\ 1, & \text{if } A = \Omega, \end{cases} \tag{8}$$

and turns out to be a completely monotone capacity (see Section 5.1 in [13]). We extend the definition allowing for $\epsilon = 0$, in which case $\nu_{p,\epsilon} = \mathbf{P}$, which stands for absence of ambiguity.

We denote by $\overline{\nu}_{p,\epsilon}$ the *dual* capacity of $\nu_{p,\epsilon}$ defined, for all $A \in \mathcal{F}$, as $\overline{\nu}_{p,\epsilon}(A) = 1 - \nu_{p,\epsilon}(A^c)$. It holds that

$$\overline{\nu}_{p,\epsilon}(A) = \begin{cases} (1-\epsilon)\mathbf{P}(A) + \epsilon, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset, \end{cases} \tag{9}$$

which is a completely alternating capacity and $\overline{\nu}_{p,\epsilon} = \max \mathcal{P}_{p,\epsilon}$.

For measuring the tastes on non-negative wealth of an investor, we consider the *power utility function* $U_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha \in (0,1)$, defined as

$$U_\alpha(x) = x^\alpha, \quad \text{for } x \geq 0. \tag{10}$$

In turn, for a non-negative $V_T \in \mathbb{R}_+^\Omega$, $p \in (0,1)$, $\epsilon \in [0,1)$ and $\alpha \in (0,1)$, this allows us to define the pessimistic and optimistic Choquet expected utilities of V_T as

$$\boldsymbol{\Phi}_{p,\epsilon,\alpha}[V_T] = \oint U_\alpha(V_T)\, \mathrm{d}\nu_{p,\epsilon} = \min_{\mathbf{P}' \in \mathcal{P}_{p,\epsilon}} \int U_\alpha(V_T)\, \mathrm{d}\mathbf{P}', \tag{11}$$

$$\overline{\boldsymbol{\Phi}}_{p,\epsilon,\alpha}[V_T] = \oint U_\alpha(V_T)\, \mathrm{d}\overline{\nu}_{p,\epsilon} = \max_{\mathbf{P}' \in \mathcal{P}_{p,\epsilon}} \int U_\alpha(V_T)\, \mathrm{d}\mathbf{P}', \tag{12}$$

that can be interpreted as lower and upper expected utilities, respectively.

In this paper our aim is to treat random wealth at maturity that can take both positive and negative values: this permits to model different behaviors, distinguishing between gains and losses.

For a random wealth at maturity $V_T \in \mathbb{R}^\Omega$, we consider its positive and negative parts $[V_T]^+ := \max\{V_T, 0\}$ and $[V_T]^- := \max\{-V_T, 0\}$. Next, for fixed $p \in (0,1)$, $\boldsymbol{\epsilon} = (\epsilon_+, \epsilon_-) \in [0,1)^2$, $\boldsymbol{\alpha} = (\alpha_+, \alpha_-) \in (0,1)^2$, and $\lambda > 0$, we define the functional

$$\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_T] = \boldsymbol{\Phi}_{p,\epsilon_+,\alpha_+}[[V_T]^+] - \lambda \overline{\boldsymbol{\Phi}}_{p,\epsilon_-,\alpha_-}[[V_T]^-]. \tag{13}$$

Such a functional is the difference of two parts that correspond to gains and losses, respectively. The first term is a lower expected utility of gains while the second term is an upper expected utility of losses which is multiplied by the scale parameter λ and subtracted. We can say that $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ realizes the (global) most pessimistic approach since it subtracts the most pessimistic (i.e., the largest) expected utility of losses to the most pessimistic (i.e., the smallest) expected utility of gains, the first scaled by parameter λ. We point out that $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ is neither concave nor convex on $\mathbb{R}^\Omega$.

Let us notice that the function $U_{\boldsymbol{\alpha},\lambda} : \mathbb{R} \to \mathbb{R}$ defined as

$$U_{\boldsymbol{\alpha},\lambda}(x) = \begin{cases} U_{\alpha_+}(x) & \text{if } x \geq 0, \\ -\lambda U_{\alpha_-}(-x) & \text{if } x < 0, \end{cases} \tag{14}$$

is an *S-shaped utility function* according to the *Cumulative Prospect Theory (CPT)* [18]. Indeed, the branch for $x \geq 0$ is concave, expressing risk aversion, while the branch for $x < 0$ is convex, expressing risk seeking. Further, the parameter λ allows to scale the impact of losses. In turn, our functional $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ can be given a CPT-like expression by recurring to $U_{\boldsymbol{\alpha},\lambda}$.

Proposition 1. *For $V_T \in \mathbb{R}^\Omega$, let σ be a permutation of Ω such that $V_T(\sigma(1)) \leq \ldots \leq V_T(\sigma(2^T))$ and i^*, j^* be indices such that $V_T(\sigma(i)) \geq 0$ for $i \geq i^*$ and $V_T(\sigma(i)) \leq 0$ for $i \leq j^*$, with $i^* := 2^T + 1$ if $V_T < 0$ and $j^* := 0$ if $V_T > 0$. Then, it holds that*

$$\begin{aligned}\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_T] = &\sum_{i=i^*}^{2^T} U_{\boldsymbol{\alpha},\lambda}(V_T(\sigma(i)))[\nu_{p,\epsilon_+}(E_\sigma^\uparrow(i)) - \nu_{p,\epsilon_+}(E_\sigma^\uparrow(i+1))] \\ &+ \sum_{i=1}^{j^*} U_{\boldsymbol{\alpha},\lambda}(V_T(\sigma(i)))[\overline{\nu}_{p,\epsilon_-}(E_\sigma^\downarrow(i)) - \overline{\nu}_{p,\epsilon_-}(E_\sigma^\downarrow(i-1))],\end{aligned}$$

where $E_\sigma^\uparrow(i) := \{\sigma(i), \ldots, \sigma(2^T)\}$, $E_\sigma^\downarrow(i) := \{\sigma(1), \ldots, \sigma(i)\}$, $E_\sigma^\downarrow(2^T + 1) := E_\sigma^\downarrow(0) := \emptyset$, and each of the two summations vanishes if $i^* = 2^T + 1$ or $j^* = 0$.

Let us stress that the difference in between our functional $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ and that appearing in classical CPT rests in how ambiguity is incorporated in the real-world probability measure $\mathbf{P}$. Indeed, in classical CPT, in place of ν_{p,ϵ_+} and $\overline{\nu}_{p,\epsilon_-}$, the authors refer to two capacities $w_+ \circ \mathbf{P}$ and $w_- \circ \mathbf{P}$, where $w_+, w_- : [0,1] \to [0,1]$ are two order automorphisms. In general, the two capacities $w_+ \circ \mathbf{P}$ and $w_- \circ \mathbf{P}$ have no particular properties, besides monotonicity with respect to set inclusion. In the particular sub-case of *rank-dependent utility models* (see, e.g., [17, 21]) the two capacities $w_+ \circ \mathbf{P}$ and $w_- \circ \mathbf{P}$ are assumed to be dual.

From a behavioral point of view, the functional $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ represents the agents' preferences over the set of final wealth V_T's. Thus, maximizing $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ we are actually choosing in agreement with the agent's preferences.

Given an initial endowment $V_0 \in \mathbb{R}$, our aim is to select a self-financing strategy $\{\theta_0, \ldots, \theta_{T-1}\}$ resulting in a final wealth $V_T \in \mathbb{R}^\Omega$, solving

$$\max_{\theta_0,\ldots,\theta_{T-1}} \boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_T]. \tag{15}$$

We notice that V_0 can be negative since is identified with the difference between the agent's initial wealth and a fixed reference wealth.

Taking into account (3), which is due to the completeness of the market, the above problem can be rewritten maximizing over the final wealth random variables V_T's that can be reached with the fixed initial endowment V_0

$$\begin{aligned}&\text{maximize } \boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_T] \text{ subject to:} \\ &\begin{cases} \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T V_0 = 0, \\ V_T \in \mathbb{R}^\Omega. \end{cases}\end{aligned} \tag{16}$$

Notice that problem (15) seeks a stochastic process $\{\theta_0, \ldots, \theta_{T-1}\}$, which is a self-financing strategy, while problem (16) looks for a random variable V_T, which is a final wealth.

Similarly to [11], problem (16) can be solved by recurring to two parametric problems that depend on the two parameters $A \in \mathcal{F}_T$ and $\eta \geq [V_0]^+$:

$$\begin{aligned} &\text{maximize } \boldsymbol{\Phi}_{p,\epsilon_+,\alpha_+}[V_T] \text{ subject to:} \\ &\begin{cases} \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T \eta = 0, \\ V_T(i) \geq 0, \quad \text{for all } i \in A, \\ V_T(i) = 0, \quad \text{for all } i \in A^c, \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\text{minimize } \overline{\boldsymbol{\Phi}}_{p,\epsilon_-,\alpha_-}[V_T] \text{ subject to:} \\ &\begin{cases} \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T (\eta - V_0) = 0, \\ V_T(i) \geq 0, \quad \text{for all } i \in A^c, \\ V_T(i) = 0, \quad \text{for all } i \in A. \end{cases} \end{aligned} \tag{18}$$

The parameter A turns out to be the set of gain states in the final wealth V_T, while A^c is the set of loss states. On the other hand, the parameter η can be interpreted as the gain level of the initial endowment V_0, while $\eta - V_0$ is the loss level. No dominance relation generally holds between optimal values of (17) and (18).

We notice that, for fixed $A \in \mathcal{F}_T$ and $\eta \geq [V_0]^+$, if $V_T^+, V_T^- \in \mathbb{R}_+^\Omega$ are optimal solutions of (17) and (18), respectively, then defining $V_T^{A,\eta} = V_T^+ - V_T^-$ we get that

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}[V_T^{A,\eta}] &= \mathbf{E}^{\mathbf{Q}}[V_T^+ - V_T^-] = \mathbf{E}^{\mathbf{Q}}[V_T^+] - \mathbf{E}^{\mathbf{Q}}[V_T^-] \\ &= (1+r)^T \eta - (1+r)^T (\eta - V_0) = (1+r)^T V_0. \end{aligned}$$

Hence, $V_T^{A,\eta}$ is a feasible solution of (16) for which $\Psi_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_T^{A,\eta}]$ is maximum, given the pair of parameters A, η that correspond to a fixed decomposition of gains and losses at final and initial times, respectively. Therefore, an optimal solution of (16) can be found by solving (17) and (18), and then maximizing by varying $A \in \mathcal{F}_T$ and $\eta \geq [V_0]^+$.

The following theorem shows that problems (17) and (18) are equivalent to two problems with non-linear constraints and an extra scalar variable.

Theorem 1. *Let $A \in \mathcal{F}_T$ and $\eta \geq [V_0]^+$. The following statements hold:*

(i) $V_T^+ \in \mathbb{R}_+^\Omega$ solves problem (17) if and only if it solves the problem

$$\begin{aligned} &\text{maximize } c \textit{ subject to:} \\ &\begin{cases} \mathbf{E}^{\mathbf{P}^i}[U_{\alpha_+}(V_T)] \geq c, \quad \textit{for all } \mathbf{P}^i \in \operatorname{ext}(\mathcal{P}_{p,\epsilon_+}), \\ \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T \eta = 0, \\ V_T(i) \geq 0, \quad \textit{for all } i \in A, \\ V_T(i) = 0, \quad \textit{for all } i \in A^c; \end{cases} \end{aligned} \tag{19}$$

(ii) $V_T^- \in \mathbb{R}_+^\Omega$ *solves problem* (18) *if and only if it solves the problem*

$$\text{minimize } c \textit{ subject to:}$$
$$\begin{cases} \mathbf{E}^{\mathbf{P}^i}[U_{\alpha_-}(V_T)] \leq c, & \textit{for all } \mathbf{P}^i \in \text{ext}(\mathcal{P}_{p,\epsilon_-}), \\ \mathbf{E}^{\mathbf{Q}}[V_T] - (1+r)^T(\eta - V_0) = 0, \\ V_T(i) \geq 0, & \textit{for all } i \in A^c, \\ V_T(i) = 0, & \textit{for all } i \in A. \end{cases} \tag{20}$$

For fixed $A \in \mathcal{F}_T$ and $\eta \geq [V_0]^+$, we let $\overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta)$ and $\underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta)$ be the optimal values of problems (17) and (18) (or, equivalently, problems (19) and (20)), respectively.

We notice that, if $A = \emptyset$ and $\eta = 0$, then (17) has only one feasible solution $V_T = 0$ and we set $\overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta) := 0$. If $A = \emptyset$ and and $\eta > 0$, then (17) has no feasible solution, therefore we set $\overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta) := -\infty$.

Similarly, if $A = \Omega$ and $\eta = V_0$, then (18) has only one feasible solution $V_T = 0$ and we set $\underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta) := 0$. If $A = \Omega$ and $\eta \neq V_0$, then (18) has no feasible solution, therefore we set $\underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta) := +\infty$.

The following theorem states that the resolution of (16) can be actually faced as a search procedure through the values $\overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta)$ and $\underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta)$.

Theorem 2. *For a final wealth* $V_T^* \in \mathbb{R}^\Omega$*, the following statements are equivalent:*

(i) V_T^* *solves problem* (16)*;*
(ii) A^* *and* $\eta^* \geq [V_0]^+$ *solve the problem*

$$\text{maximize } \overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta) - \lambda \underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta) \textit{ subject to:}$$
$$\begin{cases} A \in \mathcal{F}_T, \\ \eta \geq [V_0]^+, \\ \eta = 0, & \textit{if } A = \emptyset, \\ \eta = V_0, & \textit{if } A = \Omega, \end{cases} \tag{21}$$

and it holds

$$\overline{\psi}_{p,\epsilon_+,\alpha_+}(A^*,\eta^*) = \boldsymbol{\Phi}_{p,\epsilon_+,\alpha_+}[V_T^* \mathbf{1}_{A^*}], \tag{22}$$

$$\underline{\psi}_{p,\epsilon_-,\alpha_-}(A^*,\eta^*) = \overline{\boldsymbol{\Phi}}_{p,\epsilon_-,\alpha_-}[-V_T^* \mathbf{1}_{(A^*)^c}]. \tag{23}$$

Computationally, Theorem 2 allows to introduce the following resolution procedure. For every $A \in \mathcal{F}_T$, we look for $\eta_A \geq [V_0]^+$ that maximizes the difference $\overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta) - \lambda \underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta)$, through an iterative search algorithm. Finally, A^* and η^* are found maximizing the previous difference over all the possible pairs A, η_A.

4 A Paradigmatic Example

We consider a behavioral portfolio selection problem in a binomial market model with $S_0 = \$100$, $u = 2$, $d = \frac{1}{u}$, $r = 0$, $q = \frac{1}{3}$, $T = 2$, for which $\Omega = \{1, 2, 3, 4\}$. Figure 1 shows the recombining binomial tree of the stock price process $\{S_0, S_1, S_2\}$.

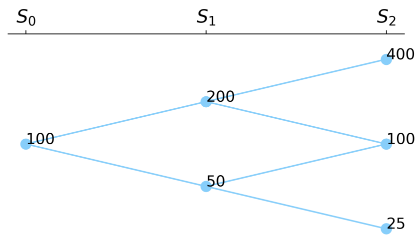

Fig. 1. Recombining binomial tree of the stock price process $\{S_0, S_1, S_2\}$.

We assume that the market agent's preferences on final wealth at time $T = 2$ are represented by the functional $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$ defined in (13) with parameters $p = \frac{1}{4}$, $\boldsymbol{\epsilon} = (0.05, 0.1)$, $\boldsymbol{\alpha} = (0.5, 0.7)$, and $\lambda = 0.4$. This market agent is quite convinced of the "real-world" probability measure $\mathbf{P}$ when concerning gains, while he/she is less convinced of it when concerning losses. Moreover, Figure 2 shows the corresponding S-shaped utility $U_{\boldsymbol{\alpha},\lambda}$ defined in (14), according to which the agent is more sensitive to gains than losses in the interval $[-2.5^5, 2.5^5]$, while the behavior changes outside of this interval.

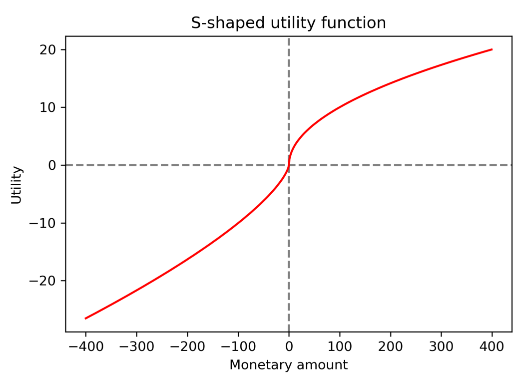

Fig. 2. S-shaped utility $U_{\boldsymbol{\alpha},\lambda}$ with parameters $\boldsymbol{\alpha} = (0.5, 0.7)$ and $\lambda = 0.4$.

We take an initial endowment $V_0 = \$5$ and find the optimal A^* and η^* in (21) through an iterative procedure based on the resolution of (19) and (20), relying on the `couenne` solver [5], which is accessed in Python through the `Pyomo` library [16]. For every $A \in \mathcal{F}_2$, we take η_A which maximizes $h^A_{V_0,r,p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}(\eta) := \overline{\psi}_{p,\epsilon_+,\alpha_+}(A,\eta) - \lambda \underline{\psi}_{p,\epsilon_-,\alpha_-}(A,\eta)$ (we denote events omitting braces and commas):

A	$\emptyset$	1	2	3	4	12	13	14
η_A	*infeasible*	5	5	5	5	49.9554	49.9554	5
$h^A_{V_0,r,p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$	$-\infty$	1.7923	0.8449	0.8449	0.3983	2.2373	2.2373	1.8361

A	23	24	34	123	124	134	234	Ω
η_A	5	5	5	65.6861	5	5	5	5
$h^A_{V_0,r,p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$	1.1949	0.9341	0.9341	2.6553	2.0211	2.0211	1.2595	2.2724

We get that $A^* = \{1,2,3\}$, $\eta^* = 65.6861$, and the optimal final wealth is

Ω	1	2	3	4
V_2^*	102.3187	45.4750	45.4750	-546.1749

for which it holds $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_2^*] = 2.6553$.

In turn, the random variable V_2^* gives rise to the process $\{V_0^*, V_1^*, V_2^*\}$ through (2) with $V_0^* = V_0 = \$5$. It is easily verified that V_2^* is a function of S_2, therefore, due to the Markov property of the process $\{S_1, S_2, S_3\}$ with respect to $\mathbf{Q}$ (see, e.g., [4]), also $\{V_0^*, V_1^*, V_2^*\}$ can be represented on a recombining binomial tree. Moreover, by (1) we derive the self-financing strategy $\{\theta_0^*, \theta_1^*\}$ that can be still represented on a recombining binomial tree. Figure 3 shows both processes $\{V_0^*, V_1^*, V_2^*\}$ and $\{\theta_0^*, \theta_1^*\}$.

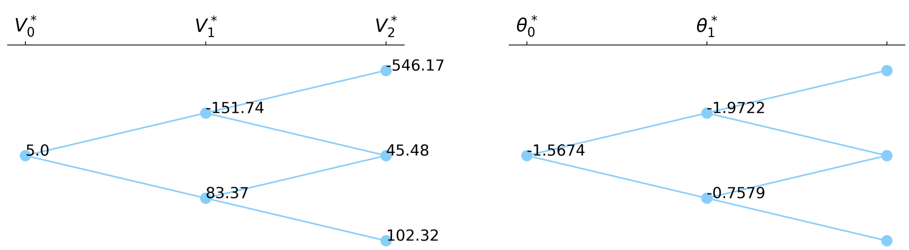

Fig. 3. Recombining binomial trees of the wealth process $\{V_0^*, V_1^*, V_2^*\}$ and the self-financing strategy $\{\theta_0^*, \theta_1^*\}$.

Next, we consider a negative endowment, by fixing $V_0 = -\$5$, for which:

A	$\emptyset$	1	2	3	4	12	13	14
η_A	0	41.9524	3.2948	3.2948	0.4730	65.3586	65.3586	31.2460
$h^A_{V_0,r,p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$	-0.8654	1.0414	-0.5475	-0.5475	-0.8901	1.6554	1.6554	0.7868

A	23	24	34	123	124	134	234	Ω
η_A	8.0296	3.3521	3.3521	80.9289	47.0364	47.0364	3.1041	*infeasible*
$h^A_{V_0,r,p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}$	-0.2409	-0.5964	-0.5964	2.0937	1.3005	1.3005	-0.8583	$-\infty$

We find the optimal $A^{**} = \{1, 2, 3\}$, $\eta^{**} = 80.9289$ and the optimal final wealth

Ω	1	2	3	4
V_T^{**}	126.0623	56.0277	56.0277	-773.3601

for which it holds $\boldsymbol{\Psi}_{p,\boldsymbol{\epsilon},\boldsymbol{\alpha},\lambda}[V_2^{**}] = 2.0937$. Also in this case, V_2^{**} is a function of S_2, therefore, following the same steps of the previous case, we derive the processes $\{V_0^{**}, V_1^{**}, V_2^{**}\}$ and $\{\theta_0^{**}, \theta_1^{**}\}$ depicted in Figure 4.

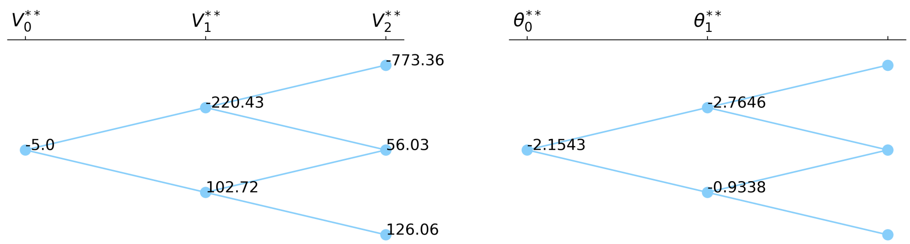

Fig. 4. Recombining binomial trees of the wealth process $\{V_0^{**}, V_1^{**}, V_2^{**}\}$ and the self-financing strategy $\{\theta_0^{**}, \theta_1^{**}\}$.

5 Conclusions

In this work we have presented a dynamic portfolio selection problem, with the aim of modeling a market agent that has different behaviors when facing gains or losses. We encoded agent's preferences in a CPT-like functional, based on an S-shaped utility function and two epsilon-contaminations of the "real-world" probability measure, related to gains and losses. Due to the completeness of the binomial market, we formulated the dynamic portfolio selection in terms of the final wealth. Next, we reduced the optimization to an iterative search problem over the set of optimal solutions of a family of non-linear optimization problems, parameterized by the set of gain states in the final wealth and the gain level of the initial endowment.

As an aim of future research, a thorough sensitivity analysis on the several parameters appearing in the model can be envisaged. Indeed, natural questions arise on the interactions of the risk parameter $\boldsymbol{\alpha}$, the ambiguity parameter $\boldsymbol{\epsilon}$, and the loss importance parameter λ. Another important line of future research concerns the study of the present model when passing to continuous time, since the underlying binomial market model is known to converge to the Black-Scholes market model [4].

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