Generating uninorms on a bounded lattice L based on uninorms on a sublattice of L

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Abstract. In this paper, we introduce two approaches for getting new classes of uninorms on bounded lattices using the indicated uninorm on a sublattice $[0,k]$ (or $[t,1]$) of a bounded lattice L with a neutral element $e \in]0,k[$ (or $e \in]t,1[$). As a by-product, these tools encompass the construction methods proposed by Çaylı (2018). Moreover, we provide some examples to assess the differences between our methods and the existing approaches.

Keywords: Bounded lattice; Neutral element; T-norm; T-conorm; Uninorm

1 Introduction

Schweizer and Sklar [34] introduced the notion of triangular norms (t-norms, for short) in their present form based on some ideas of Menger [27] devoted to generalizing the triangle inequality in metric spaces to probabilistic metrics spaces. The dual notion of a t-norm is a triangular conorm (t-conorm, for short) introduced in [34]. These operators play substantial roles in many fields, such as fuzzy systems modeling, fuzzy logic, fuzzy set theory, probabilistic metric spaces, decision-making, approximate reasoning, and information aggregation [1, 2, 16, 23–25, 28].

Uninorms on the unit interval that are important generalizations of t-norms and t-conorms were introduced by Yager and Rybalov [40]. These operators allow the neutral element e to locate anywhere in $[0,1]$. In particular, a uninorm degenerates to a t-norm (or t-conorm) when $e = 1$ (or $e = 0$). A uninorm with a neutral element $e \in]0,1[$ is usually called proper. They have become significant theoretical subjects [13–15, 18] and indispensable application fields, such as decision-making [39], neural networks [3], fuzzy system modeling [38], and fuzzy logic, in general [31].

Uninorms on bounded lattices introduced by Karaçal and Mesiar [22] have drawn much attention from researchers. In particular, they showed the existence of a uninorm on a bounded lattice L with a neutral element $e \in L\backslash\{0,1\}$ with the underlying t-norms or t-conorms. Since then, these operators have been investigated in detail on account of the fact that bounded lattices are more general structures than the unit interval. Bodjanova and Kalina [6] constructed uninorms on L simultaneously based on t-norms and t-conorms. Subsequently, Çaylı [7, 8] presented some construction methods for uninorms on bounded lattices with the help of t-norms and t-conorms under certain additional assumptions. Several construction approaches for uninorms on bounded lattices can be found in the literature, including the ones by using t-norms (or t-conorms) [5, 11, 12, 35, 36], closure (or interior) operators [10, 32, 42], t-subnorms (or t-superconorms) [20, 21, 41], and additive generators [19].

The construction of uninorms on bounded lattices is a compelling issue because of the complicated structures of bounded lattices compared to the unit interval. Many existing papers dealing with uninorms on bounded lattices focussed on their generations based on t-norms or t-conorms. In recent years, Xiu and Zheng [37] have provided a method for obtaining uninorms via uninorms on a sublattice of a bounded lattice. Although their way brings a new perspective to discuss the constructions of uninorms on bounded lattices, these operators have not been characterized precisely yet. In this paper, we propose new approaches for generating uninorms on a bounded lattice L by considering the existence of a uninorm on a sublattice of L instead of t-norms and t-conorms as in many known constructions. Namely, we extend construction methods for uninorms via t-norms (or t-conorms) to those via uninorms defined on a sublattice of L. Since uninorms are more general than t-norms and t-conorms, our methods based on the existence of a uninorm are more effective than those in the literature generated by t-norms or t-conorms. It is worth noting that some known methods to obtain uninorms on bounded lattices can be derived from our tools. Hence, our results can be of help to the enrichment of the class of uninorms on bounded lattices and their characterization. Furthermore, from the theoretical point of view, our tools are an efficient contribution to the research subject based on uninorms on sublattices of L.

The remainder of this paper is organized as follows: In Section 2, we recall some preliminary details about uninorms on bounded lattices. In Section 3, we extend the fixed uninorm on a sublattice $[0,k]$ (or $[t,1]$) of a bounded lattice L to a new uninorm on L with a neutral element $e \in]0,k[$ (or $e \in]t,1[$). In this case, our tools present new classes of uninorms on bounded lattices and generalize the construction methods introduced in [7]. Some specific examples are also provided to illustrate that new methods differ from construction methods for uninorms presented in [7, 37]. In the final section, some concluding remarks are added.

2 Preliminaries

In this section, we recall some basic notions and results related to lattices and uninorms on bounded lattices.

A lattice [4] is a nonempty set L equipped with a partial order $\leq$ such that any two elements $x, y \in L$ have a smallest upper bound (called join or supremum), written as $x \vee y$, as well as a greatest lower bound (called meet or infimum), written as $x \wedge y$. For $x, y \in L$, the symbol $x < y$ means that $x \leq y$ and $x \neq y$. The elements x and y are comparable, denoted by $x \nparallel y$, if $x \leq y$ or $y < x$. Otherwise, they are incomparable, in this case we use the notation $x \parallel y$.

In the following, for the elements $x, y \in L$, the set of all elements incomparable with x is denoted by I_x, i.e., $I_x = \{z \in L : z \parallel x\}$. The set of all elements incomparable with both x and y is denoted by $I_{x,y}$, i.e., $I_{x,y} = \{z \in L : z \parallel x \text{ and } z \parallel y\}$. I_x^y denotes the set of all elements that are incomparable with x but comparable with y, i.e., $I_x^y = \{z \in L : z \parallel x$ and $z \nparallel y\}$

A lattice $(L,\leq,\wedge,\vee)$ is called bounded if it has a bottom element and a top element, written as 0 and 1, respectively. Throughout this paper, unless stated otherwise, L denotes a bounded lattice $(L,\leq,\wedge,\vee)$ with the bottom element 0 and the top element 1.

Let $(L,\leq,\wedge,\vee)$ be a lattice, $x,y\in L$ and $x\leq y$. The subinterval $[x,y]$ of L is defined such that

$$[x,y]=\{z\in L: x\leq z\leq y\}.$$

Other subintervals such as $[x,y[$, $]x,y]$, and $]x,y[$ of L can be defined similarly. Obviously, $([x,y],\leq,\wedge,\vee)$ is a bounded lattice with the bottom element x and the top element y.

Definition 1 ([22]). *A binary operation $U: L\times L\to L$ is called a uninorm if, for any $x,y,z\in L$, it satisfies the following properties:*
(i) $U(x,z)\leq U(y,z)$ for $x\leq y$ (increasingness);
(ii) $U(x,U(y,z))=U(U(x,y),z)$ (associativity);
(iii) $U(x,y)=U(y,x)$ (commutativity);
(iv) there is an element $e\in L$, called a neutral element of U, such that $U(x,e)=x$ for all $x\in L$ (neutral element).

A uninorm U on L is called idempotent if $U(x,x)=x$ for all $x\in L$. A uninorm U on L is called conjunctive (resp. disjunctive) if $U(0,1)=0$ (resp. $U(0,1)=1$). Obviously, a t-norm T (resp. t-conorm S) on L is exactly a uninorm U on L with the neutral element $e=1$ (resp. $e=0$) (see [9, 17, 26, 29, 30, 33]).

Example 1. (i) The largest t-norm $T_\wedge$ on $[z_1,z_2]^2$ is defined by $T_\wedge(x,y)=x\wedge y$ for all $x,y\in[z_1,z_2]$, while the smallest t-norm T_W on $[z_1,z_2]^2$ puts the value of $x\wedge y$ if $z_2\in\{x,y\}$ and z_1 otherwise. Thus, for any t-norm T on $[z_1,z_2]^2$, there holds $T_W\leq T\leq T_\wedge$.

(ii) The smallest t-conorm $S_\vee$ on $[z_1,z_2]^2$ is defined by $S_\vee(x,y)=x\vee y$ for all $x,y\in[z_1,z_2]$, while the largest t-conorm S_W on $[z_1,z_2]^2$ puts the value $x\vee y$ if $z_1\in\{x,y\}$ and z_2 otherwise. Thus, for any t-conorm S on $[z_1,z_2]^2$, there holds $S_\vee\leq S\leq S_W$.

Proposition 1 ([22]). *Let U be a uninorm on L with the neutral element $e\in L\backslash\{0,1\}$. Then the following results hold:*
(i) $U\mid[0,e]^2:[0,e]^2\to[0,e]$ is a t-norm on $[0,e]^2$.
(ii) $U\mid[e,1]^2:[e,1]^2\to[e,1]$ is a t-conorm on $[e,1]^2$.

Definition 2 ([41]). *Let $e\in L\backslash\{0,1\}$. $\mathscr{U}_{min}$ denotes the class of all uninorms U on L with the neutral element e satisfying the following condition:*

$$U(x,y)=y,\ \textit{for all}\ (x,y)\in\,]e,1]\times L\backslash[e,1].$$

Similarly, $\mathscr{U}_{max}$ denotes the class of all uninorms U on L with the neutral element e satisfying the following condition:

$$U(x,y)=y,\ \textit{for all}\ (x,y)\in[0,e[\,\times L\backslash[0,e].$$

3 Generating uninorms on bounded lattices

Following the construction of uninorms on the unit interval, their definition and construction related to algebraic structures on bounded lattices have become an attractive research area in recent years. Karaçal and Mesiar [22] introduced the concept of uninorms on bounded lattices and showed the existence of such uninorms via t-norms or t-conorms. Recently, Xiu and Zheng [37] described a construction method for uninorms with the help of a uninorm on a subinterval of a bounded lattice. In this section, we provide new approaches to obtain uninorms on a bounded lattice L via a uninorm defined on the subinterval $[0,k]$ (resp. $[t,1]$) of L with a neutral element $e \in]0,k[$ (resp. $e \in]t,1[$) under some additional assumptions. As a special case, these methods encompass the constructions of uninorms introduced in [7]. Moreover, we give some illustrative examples to show that these approaches differ from the ones previously presented in [7, 37].

Theorem 1. *Let $k \in L \backslash \{0,1\}$, $U' : [0,k]^2 \to [0,k]$ be a uninorm with a neutral element $e \in]0,k[$ and the function $U_L : L^2 \to L$ be given by the formula (1).*

$$U_L(x,y) = \begin{cases} U'(x,y) & if\ (x,y) \in [0,k]^2, \\ y & if\ (x,y) \in [0,e] \times (I_k \cup]k,1]), \\ x & if\ (x,y) \in (I_k \cup]k,1]) \times [0,e], \\ x \vee y \vee k & if\ (x,y) \in I_k \times I_k, \\ 1 & otherwise. \end{cases} \tag{1}$$

(I) If $a \| b$ for all $a \in I_k$, $b \in I_e^k \cup]e,k]$ and $U'(x,y) \in [0,e]$ implies $x,y \in [0,e]$, then U_L is a uninorm on L with a neutral element e.

(II) If $c \vee k \in]k,1[$ for all $c \in I_k$ and $U'(x,y) \in [0,e]$ implies $x,y \in [0,e]$, then U_L is a uninorm on L with a neutral element e if and only if $a \| b$ for all $a \in I_k$, $b \in I_e^k \cup]e,k]$.

(III) If there is an element $d \in I_k$ and $c \vee k \in]k,1[$ for all $c \in I_k$, then U_L is a uninorm on L with a neutral element e if and only if $a \| b$ for all $a \in I_k$, $b \in I_e^k \cup]e,k]$ and $U'(x,y) \in [0,e]$ implies $x,y \in [0,e]$.

Remark 1. The structure of uninorm $U_L : L^2 \to L$ in Theorem 1 is shown in Figure 1.

In the following example, we illustrate that the condition that $U'(a,b) \in [0,e]$ implies $a,b \in [0,e]$ is not necessary for the function U_L given by the formula (1) to yield a uninorm on a bounded lattice L.

Example 2. Consider the bounded lattice L_1 depicted by Hasse diagram in Figure 3 and the function $U' : [0,k]^2 \to [0,k]$ defined in Table 1. It is easy to see that U' is a uninorm on $[0,k]^2$ with the neutral element e. In addition, $U'(a,b) \in [0,e]$ for all $(a,b) \in [0,e[\times \left([e,k] \cup I_e^k\right)$. By using the construction approach in Theorem 1, we define the function U_L on L_1 in Table 2. Moreover, it is a uninorm on L_1 with the neutral element e.

We should also point out that the condition that $a \| b$ for all $a \in I_k$, $b \in I_e^k \cup]e,k]$ is not necessary for the function U_L given by the formula (1) to generate a uninorm on a bounded lattice L. By the following example, we demonstrate this fact.

I_k	y	1	1	1	$x \vee y \vee k$
I_e^k	$U'(x,y)$	$U'(x,y)$	1	$U'(x,y)$	1
$]k,1]$	y	1	1	1	1
$]e,k]$	$U'(x,y)$	$U'(x,y)$	1	$U'(x,y)$	1
$[0,e]$	$U'(x,y)$	$U'(x,y)$	x	$U'(x,y)$	x
	$[0,e]$	$]e,k]$	$]k,1]$	I_e^k	I_k

Fig. 1: Uninorm $U_L : L^2 \to L$ in Theorem 1

I_t	0	0	y	0	$x \wedge y \wedge t$
I_e^t	0	$U''(x,y)$	$U''(x,y)$	$U''(x,y)$	0
$[e,1]$	x	$U''(x,y)$	$U''(x,y)$	$U''(x,y)$	x
$[t,e[$	0	$U''(x,y)$	$U''(x,y)$	$U''(x,y)$	0
$[0,t[$	0	0	y	0	0
	$[0,t[$	$[t,e[$	$[e,1]$	I_e^t	I_t

Fig. 2: Uninorm $U^L : L^2 \to L$ in Theorem 2

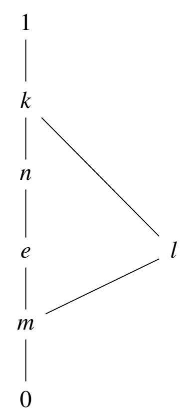

Fig. 3: The lattice L_1

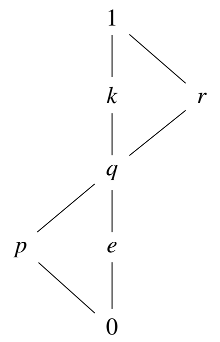

Fig. 4: The lattice L_2

Table 1: The uninorm U' on $[0,k]^2$

U'	0	m	l	e	n	k
0	0	0	0	0	0	0
m	0	m	m	m	m	m
l	0	m	l	l	l	l
e	0	m	l	e	n	k
n	0	m	l	n	k	k
k	0	m	l	k	k	k

Table 2: The uninorm U_L in L_1

U_L	0	m	l	e	n	k	1
0	0	0	0	0	0	0	1
m	0	m	m	m	m	m	1
l	0	m	l	l	l	l	1
e	0	m	l	e	n	k	1
n	0	m	l	n	k	k	1
k	0	m	l	k	k	k	1
1	1	1	1	1	1	1	1

Table 3: The uninorm U' on $[0,k]^2$

U'	0	e	p	q	k
0	0	0	p	q	k
e	0	e	p	q	k
p	p	p	k	k	k
q	q	q	k	k	k
k	k	k	k	k	k

Table 4: The uninorm U_L in L_2

U_L	0	e	p	q	k	r	1
0	0	0	p	q	k	r	1
e	0	e	p	q	k	r	1
p	p	p	k	k	k	1	1
q	q	q	k	k	k	1	1
k	k	k	k	k	k	1	1
r	r	r	1	1	1	1	1
1	1	1	1	1	1	1	1

Example 3. Consider the bounded lattice L_2 characterized by Hasse diagram in Figure 4 and the uninorm $U' : [0,k]^2 \to [0,k]$ given as in Table 3. Obviously, $p,q < r$ for the elements $p \in I_e^k$, $q \in]e,k]$ and $r \in I_k$. If applying the construction approach in Theorem 1, the function U_L on L_2 is defined by Table 4. We can immediately observe that it is a uninorm on L_2 with the neutral element e.

Remark 2. (i) The condition that $a \| b$ for all $a \in I_k$, $b \in I_e^k \cup]e,k]$ in Theorem 1 cannot be omitted, in general. Suppose that there exist $a \in I_k$, $b \in I_e^k \cup]e,k]$ such that $b < a$. If $a \vee k \in]k,1[$, we have $U_L(b,a) = 1 > a \vee k = U_L(a,a)$. So, the function U_L does not satisfy the monotonicity.
(ii) The condition that $U'(x,y) \in [0,e]$ implies $x,y \in [0,e]$ in Theorem 1 is indispensable if $L \backslash ([0,k] \cup \{1\}) \neq \emptyset$. Indeed, in the opposite case there exist $x \in [0,e] \cup I_e^k$ and $y \in]e,k] \cup I_e^k$ such that $U'(x,y) \in [0,e]$ and for $z \in]k,1] \cup I_k$ such that $z \neq 1$, we obtain $U_L(z, U_L(y,x)) = U_L(z, U'(y,x)) = z$ and $U_L(U_L(z,y),x) = U_L(1,x) = 1$. So, the function U_L does not satisfy the associativity.

If we put $e = k$ in Theorem 1, then U' is a t-norm on $[0,e]$. In this case the condition that $U'(a,b) \in [0,e]$ implies $a,b \in [0,e]$ is satisfied. In the following, we have a structure of a uninorm on L on the basis of Theorem 1, which coincides with the uninorm U_e^T introduced in [7, Theorem 2.23].

Corollary 1. *Let $e \in L \backslash \{0,1\}$ and $T : [0,e]^2 \to [0,e]$ be a t-norm. Then, the function $U_L^1 : L^2 \to L$, given by the formula (2), is a uninorm on L with the neutral element e.*

$$U_L^1(x,y) = \begin{cases} T(x,y) & if\ (x,y) \in [0,e]^2, \\ y & if\ (x,y) \in [0,e] \times (I_e \cup]e,1]), \\ x & if\ (x,y) \in (I_e \cup]e,1]) \times [0,e], \\ x \vee y \vee e & if\ (x,y) \in I_e \times I_e, \\ 1 & otherwise. \end{cases} \tag{2}$$

If we put $e = 0$ in Theorem 1, then U' is a t-conorm on $[0,k]$. In this case the condition that $U'(a,b) \in [0,e]$ implies $a,b \in [0,e]$ is satisfied. Because if $U'(a,b) = 0$, then $a = b = 0$. In the following, we have a structure of a t-conorm on L on the basis of Theorem 1.

Corollary 2. *Let $k \in L \backslash \{0,1\}$ such that $a \| b$ for all $a \in I_k$, $b \in]0,k]$. If $S : [0,k]^2 \to [0,k]$ is a t-conorm, then the function $U_L^2 : L^2 \to L$, given by the formula (3), is a t-conorm on L.*

$$U_L^2(x,y) = \begin{cases} S(x,y) & if\ (x,y) \in [0,k]^2, \\ y & if\ (x,y) \in \{0\} \times (]k,1] \cup I_k), \\ x & if\ (x,y) \in (]k,1] \cup I_k) \times \{0\}, \\ x \vee y \vee k & if\ (x,y) \in I_k \times I_k, \\ 1 & otherwise. \end{cases} \tag{3}$$

Example 4. Consider the bounded lattice L_3 characterized by Hasse diagram in Figure 5 and the function $U' : [0,k]^2 \to [0,k]$ defined by Table 5. It is obvious that U' is a uninorm on $[0,k]^2$ with the neutral element e such that $U'(a,b) \in [0,e]$ implies $a,b \in [0,e]$. If we utilize the construction approach in Theorem 1, we obtain the uninorm U_L on L_3 given as in Table 6.

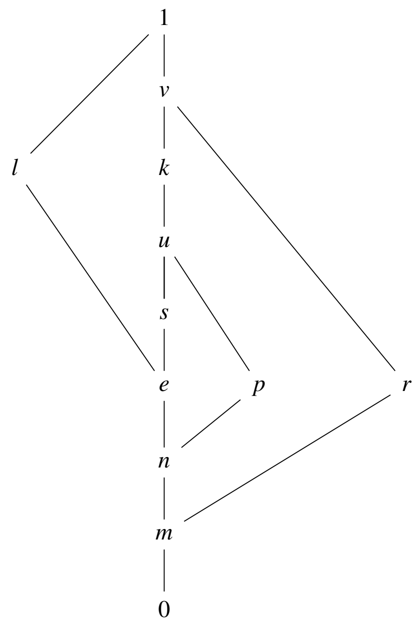

Fig. 5: Lattice L_3

It should be pointed out that the uninorm U_L given by the formula (1) in Theorem 1 does not need to coincide with the uninorms introduced in [7, Theorem 2.23] and [37, Theorem 3.1]. In the following, we give an example to illustrate this fact.

Example 5. We still consider the lattice L_3 drawn in Figure 5 and the uninorm U' on $[0,k]^2$ given in Table 5.

Table 5: The uninorm U' on $[0,k]^2$

U'	0	m	n	p	e	s	u	k
0	0	0	0	p	0	s	u	k
m	0	m	m	p	m	s	u	k
n	0	m	n	p	n	s	u	k
p	p	p	p	p	p	u	u	k
e	0	m	n	p	e	s	u	k
s	s	s	s	u	s	s	u	k
u	u	u	u	u	u	u	u	k
k	k	k	k	k	k	k	k	k

Table 6: The uninorm U_L in L_3

U_L	0	m	n	p	e	s	u	k	l	r	v	1
0	0	0	0	p	0	s	u	k	l	r	v	1
m	0	m	m	p	m	s	u	k	l	r	v	1
n	0	m	n	p	n	s	u	k	l	r	v	1
p	p	p	p	p	p	u	u	k	1	1	1	1
e	0	m	n	p	e	s	u	k	l	r	v	1
s	s	s	s	u	s	s	u	k	1	1	1	1
u	u	u	u	u	u	u	u	k	1	1	1	1
k	k	k	k	k	k	k	k	k	1	1	1	1
l	l	l	l	1	l	1	1	1	1	1	1	1
r	r	r	r	1	r	1	1	1	1	v	1	1
v	v	v	v	1	v	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1

(i) Based on the construction method in [37, Theorem 3.1], the uninorm U_1 on L_3 is shown in Table 7, where the t-superconorm $R : [k,1]^2 \to [k,1]$ is defined $R(x,y) = x \vee y$ for all $x,y \in [k,1]$. Obviously, $U_L(r,s) = 1 \neq v = U_1(r,s)$. Therefore, U_L is different from U_1 on L_3. Indeed, we observe that regardless of the choice of the t-superconorm R, uninorms generated by these two methods does not coincide on L_3. By the method in Theorem 1, we have $U_L(r,r) = v$, $U_L(r,s) = 1$ (i.e, $U_L(r,s) > U_L(r,r)$), while we obtain $U_1(r,r) = R(r \vee k, r \vee k) = R(v,v)$ and $U_1(r,s) = R(r \vee k, s \vee k) = R(v,k)$ by the method in [37, Theorem 3.1]. For $k < v$, by the monotonicity of the t-superconorm R, $R(v,k) \leq R(v,v)$, i.e., $U_1(r,s) \leq U_1(r,r)$. Hence, there is no such t-superconorm R that uninorms generated by these two methods coincide on L_3.

Table 7: The uninorm U_1 on L_3

U_1	0	m	n	p	e	s	u	k	l	r	v	1
0	0	0	0	p	0	s	u	k	l	r	v	1
m	0	m	m	p	m	s	u	k	l	r	v	1
n	0	m	n	p	n	s	u	k	l	r	v	1
p	p	p	p	p	p	u	u	k	1	v	v	1
e	0	m	n	p	e	s	u	k	l	r	v	1
s	s	s	s	u	s	s	u	k	1	v	v	1
u	u	u	u	u	u	u	u	k	1	v	v	1
k	k	k	k	k	k	k	k	k	1	v	v	1
l	l	l	l	1	l	1	1	1	1	1	1	1
r	r	r	r	v	r	v	v	v	1	v	v	1
v	v	v	v	v	v	v	v	v	1	v	v	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Table 8: The uninorm U_e^T on L_3

U_e^T	0	m	n	p	e	s	u	k	l	r	v	1
0	0	0	0	p	0	s	u	k	l	r	v	1
m	0	m	m	p	m	s	u	k	l	r	v	1
n	0	m	n	p	n	s	u	k	l	r	v	1
p	p	p	p	u	p	1	1	1	1	v	1	1
e	0	m	n	p	e	s	u	k	l	r	v	1
s	s	s	s	1	s	1	1	1	1	1	1	1
u	u	u	u	1	u	1	1	1	1	1	1	1
k	k	k	k	1	k	1	1	1	1	1	1	1
l	l	l	l	1	l	1	1	1	1	1	1	1
r	r	r	r	v	r	1	1	1	1	v	1	1
v	v	v	v	1	v	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1

(ii) Based on the construction method in [7, Theorem 2.23], the uninorm U_e^T on L_3 is shown in Table 8 when putting the t-norm $T_\wedge : [0,e]^2 \to [0,e]$. Obviously, $U_L(p,r) =$

$1 \neq v = U_e^T(p,r)$. Therefore, U_L is different from U_e^T on L_3. Indeed, we observe that regardless of the choice of t-norm T, uninorms generated by these two methods does not coincide on L_3. By the method in [7, Theorem 2.23], for any elements $x,y \in I_e$, we have $U_e^T(x,y) = x \vee y \vee e$. So, for $p,r \in I_e$, $U_e^T(p,r) = p \vee r \vee e = v$.

Remark 3. Consider the uninorm $U_L : L^2 \to L$ given by the formula (1) in Theorem 1. In this case, we have the following statements:
(1) U_L is a disjunctive uninorm, i.e., $U_L(0,1) = 1$.
(2) If $k = 1$, then $U_L = U'$.
(3) $U_L \in \mathscr{U}_{max}$ if and only if $U' \in \mathscr{U}_{max}$.
(4) U_L is not an idempotent uninorm on L, in general. To be more precise, for any elements $f \in I_k$ and $g \in]k,1[$, we obtain $U_L(f,f) = f \vee k \neq f$ and $U_L(g,g) = 1 \neq g$. However, we can state that U_L is an idempotent uninorm on L if and only if $I_k \cup]k,1[= \emptyset$ and U' is an idempotent uninorm on $[0,k]^2$.

In the following, by considering the existence of a uninorm on a sublattice $[t,1]$ of a bounded lattice L, we introduce a dual construction of uninorms on L with a neutral element $e \in L \backslash \{0,1\}$.

Theorem 2. *Let $t \in L \backslash \{0,1\}$, $U'' : [t,1]^2 \to [t,1]$ be a uninorm with a neutral element $e \in]t,1[$ and the function $U^L : L^2 \to L$ be given by the formula (4).*

$$U^L(x,y) = \begin{cases} U''(x,y) & if\ (x,y) \in [t,1]^2, \\ y & if\ (x,y) \in [e,1] \times (I_t \cup [0,t[), \\ x & if\ (x,y) \in (I_t \cup [0,t[) \times [e,1], \\ x \wedge y \wedge t & if\ (x,y) \in I_t \times I_t, \\ 0 & otherwise. \end{cases} \tag{4}$$

(I) If $a \| b$ for all $a \in I_t$, $b \in I_e^t \cup [t,e[$ and $U''(x,y) \in [e,1]$ implies $x,y \in [e,1]$, then U^L is a uninorm on L with a neutral element e.

(II) If $c \wedge t \in]0,t[$ for all $c \in I_t$ and $U''(x,y) \in [e,1]$ implies $x,y \in [e,1]$, then U^L is a uninorm on L with a neutral element e if and only if $a \| b$ for all $a \in I_t$, $b \in I_e^t \cup [t,e[$.

(III) If there is an element $d \in I_t$ and $c \wedge t \in]0,t[$ for all $c \in I_t$, then U^L is a uninorm on L with a neutral element e if and only if $a \| b$ for all $a \in I_t$, $b \in I_e^t \cup [t,e[$ and $U''(x,y) \in [e,1]$ implies $x,y \in [e,1]$.

Remark 4. The structure of uninorm $U^L : L^2 \to L$ in Theorem 2 is shown in Figure 2.

If we put $e = t$ in Theorem 2, then U'' is a t-conorm on $[e,1]$. In this case the condition that $U''(a,b) \in [e,1]$ implies $a,b \in [e,1]$ is satisfied. In the following, we have a structure of a uninorm on L on the basis of Theorem 2, which coincides with the uninorm U_e^S introduced in [7, Theorem 2.23].

Corollary 3. *Let $e \in L \backslash \{0,1\}$ and $S : [e,1]^2 \to [e,1]$ be a t-conorm. Then the function $U_1^L : L^2 \to L$, given by the formula (5), is a uninorm on L with the neutral element e.*

$$U_1^L(x,y) = \begin{cases} S(x,y) & if\ (x,y) \in [e,1]^2, \\ y & if\ (x,y) \in [e,1] \times (I_e \cup [0,e[), \\ x & if\ (x,y) \in (I_e \cup [0,e[) \times [e,1], \\ x \wedge y \wedge e & if\ (x,y) \in I_e \times I_e, \\ 0 & otherwise. \end{cases} \tag{5}$$

If we put $e=1$ in Theorem 2, then U'' is a t-norm on $[t,1]$. In this case the condition that $U''(a,b)\in[e,1]$ implies $a,b\in[e,1]$ is satisfied. Because if $U''(a,b)=1$, then $a=b=1$. In the following, we have a structure of a t-norm on L on the basis of Theorem 2.

Corollary 4. *Let $t\in L\backslash\{0,1\}$ such that $a\|b$ for all $a\in I_t$, $b\in[t,1[$. If $T:[t,1]^2\to[t,1]$ is a t-norm, then the function $U_2^L:L^2\to L$, given by the formula (6), is a t-norm on L.*

$$U_2^L(x,y)=\begin{cases}T(x,y) & if\ (x,y)\in[t,1]^2,\\ y & if\ (x,y)\in\{1\}\times([0,t[\cup I_t),\\ x & if\ (x,y)\in([0,t[\cup I_t)\times\{1\},\\ x\wedge y\wedge t & if\ (x,y)\in I_t\times I_t,\\ 0 & otherwise.\end{cases}\tag{6}$$

Remark 5. Consider the uninorm $U^L:L^2\to L$ given by the formula (4) in Theorem 2. In this case, we have the following statements:
(1) U^L is a conjunctive uninorm, i.e., $U^L(0,1)=0$.
(2) If $t=0$, then $U^L=U''$.
(3) $U^L\in\mathscr{U}_{min}$ if and only if $U''\in\mathscr{U}_{min}$.
(4) U^L is not an idempotent uninorm on L, in general. To be more precise, for any elements $f\in I_t$ and $g\in]0,t[$, we obtain $U_L(f,f)=f\wedge t\neq f$ and $U_L(g,g)=0\neq g$. However, we can state that U^L is an idempotent uninorm on L if and only if $I_t\cup]0,t[=\emptyset$ and U'' is an idempotent uninorm on $[t,1]^2$.

4 Concluding remarks

Uninorms on bounded lattices have been comprehensively discussed by researchers in a manner similar to their investigations on the unit interval. In particular, Xiu and Zheng [37] have recently introduced some methods for getting uninorms via uninorms on a sublattice of a bounded lattice. In this paper, we have continued to study the subject of uninorms on bounded lattices from a mathematical point of view. We have presented in Theorem 1 (resp. Theorem 2) an effective method to build a new family of disjunctive (resp. conjunctive) uninorms on a bounded lattice L. This method exploits the fixed uninorm on the sublattice $[0,k]$ (resp. $[t,1]$) of L with a neutral element $e\in]0,k[$ (resp. $e\in]t,1[$). As a by-product of Theorems 1 and 2, when putting $e=k$ or $e=t$, we have obtained the uninorms introduced in [7]. We also have proposed two classes of t-conorms and t-norms on L by taking $e=0$ in Theorem 1 and $e=1$ in Theorem 2. Moreover, we have provided Example 5 to illustrate that our construction method for uninorms differs from those introduced in [7, 37].

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