Division of Interactive Fuzzy Numbers Using Joint Possibility Distributions*

Zahra Alijani[1][0000-0002-1448-9068] and Petra Števuliáková[1][0000-0002-7879-1397]

University of Ostrava, Centre of Excellence IT4Innovations
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, Ostrava, Czech Republic
zahra.alijani@osu.cz, petra.stevuliakova@osu.cz

Abstract. This article introduces a method to calculate the division of two interactive positive fuzzy numbers. Specifically, we focus on arithmetic operations derived from completely correlated fuzzy numbers. Our study demonstrates that the proposed division yields fuzzy numbers with smaller widths compared to any other division method for positive fuzzy numbers, as determined by joint possibility distributions. Furthermore, we characterize these operations using α-cuts, providing insights into their behavior across different α-cuts. Furthermore, we establish connections between our proposed division method and existing techniques, such as the generalized Hukuhara division and the division based on Zadeh's extension principle. Finally, we provide illustrative examples to demonstrate the practical implications of our proposed method.

Keywords: Interactive Fuzzy Numbers · Generalized Hukuhara Division· J-division · Completely Correlated Fuzzy Numbers.

1 Introduction

At the heart of fuzzy logic are fuzzy sets and fuzzy numbers that enable the representation and manipulation of vague or ambiguous information. An essential aspect of fuzzy logic is the characterization of joint possibility distributions (JPD) [11], which describe the concurrent attainable states of multiple fuzzy numbers. The concept of interactivity between fuzzy numbers, originally introduced by Zadeh [23], offers a powerful framework to handle uncertainty in various domains. These two notions, joint possibility distribution and interactivity, were later associated in [5, 16] in order to understand the interactivity between fuzzy numbers by describing the simultaneously achievable states of the variables given by the joint possibility distribution.

* The study described is from the project Research of Excellence on Digital Technologies and Wellbeing CZ.02.01.01/00/22_008/0004583 which is co-financed by the European Union. The research is financially supported by the EU under the REFRESH – Research Excellence For REgion Sustainability and High-tech Industries project No. CZ.10.03.01/00/22_003/0000048 via the Operational Programme Just Transitions.

Fuzzy numbers can exhibit different levels of interactivity; in particular, for the joint possibility distribution defined by a t-norm, the properties of addition, subtraction, and multiplication of interactive fuzzy numbers have been extensively investigated in the literature [12, 15, 17, 18]. It is well established that in such cases, the joint possibility distribution is directly and pointwise defined from the membership values of its marginal possibility distributions by an aggregation operator, such as a t-norm. It is worth noting that the interactivity relation between fuzzy numbers may be governed by a more general joint possibility distribution that cannot be defined by any aggregation operator.

This paper explores joint possibility distributions based on the complete correlation between fuzzy numbers [5, 16] which represents a significant case of interactivity in which fuzzy numbers are tightly connected through a linear relationship. This concept was already studied with respect to applications; for example, in [2] the interactivity of fuzzy numbers based on complete correlation was used to represent the relationship between susceptible and infected individuals for modeling epidemic spread. In [1], initial values of a fuzzy autonomous system that solves a predato-pray population model were assumed to be completely correlated fuzzy intervals. The interactivity by means of completely correlated fuzzy numbers was then studied in the context of fuzzy differential equations with interactive parameters and initial conditions in [4]. The authors also discussed the SI-epidemiological model in two forms: susceptible and infected individuals are completely correlated, and the transfer rate and initial conditions are completely correlated.

Although the interactivity based on joint possibility distribution is quite well researched and basic arithmetic operations such as addition, subtraction, and multiplication were established, the division of interactive fuzzy numbers lacks active research, suggesting a gap in understanding this operation's behavior and implications. However, the ability to invert addition and multiplication operations is of great importance in interval and fuzzy analysis. This capability facilitates the solution of equations, handling interval and fuzzy differential equations, and conducting regression analysis effectively. Motivated by the work mentioned above and the lack of a division principle for interactive fuzzy numbers, we investigate the joint possibility distribution with respect to the division operation, which facilitates the extension of interactive fuzzy numbers. This principle accounts for the interactivity between fuzzy numbers, allowing more nuanced and context-aware calculations in fuzzy logic systems in practical applications, including areas like managing financial risks, making decisions based on multiple criteria [2, 24].

We begin by elucidating the definition and properties of JPDs, highlighting their role in representing the collective uncertainty associated with multiple fuzzy numbers. In particular, we investigate interactivity based on complete correlation and define the division of complete correlated fuzzy numbers (J_c-division) in order to find alternatives or competitors for current divisions such as generalized Hukuhara division (g-division) [25] and division based on Zadeh's extension principle. However, the division of interactive fuzzy numbers exhibits more complex behavior and requires specific approaches for its analysis. In this present work, we restrict our interest to J_c-division of positive fuzzy numbers because our focus is on the application to difference equations where only positive fuzzy numbers are considered. Our next work is an extension of the general case and other applications.

1.1 Fuzzy Notations

In this section, we give only the definition of fuzzy numbers and α-cuts and refer the other notation and definitions to [3, 10, 20]. Fuzzy numbers are fuzzy sets that are normal, fuzzy convex, upper semicontinuous, and do have compact supports, see the exact definition below.

Definition 1. *[3] Consider a fuzzy subset of the real line $A : \mathbb{R} \to [0, 1]$. Then A is a fuzzy number if it satisfies the following properties.*

(i) A is normal, that is, $\exists x_0 \in \mathbb{R}$ with $A(x_0) = 1$,
(ii) A is fuzzy convex, that is, $A(tx + (1 - t)y) \geq \min\{A(x), A(y)\}$, $\forall t \in [0, 1]$, $x, y \in \mathbb{R}$,
(iii) A is upper semicontinuous on $\mathbb{R}$ (that is, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $A(x) - A(x_0) < \varepsilon, |x - x_0| < \delta$),
(iv) A is compactly supported, that is, $\overline{\{x \in \mathbb{R}; A(x) > 0\}}$ is compact.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ fuzzy numbers in $\mathbb{R}$ and by $\mathbb{R}^+_{\mathcal{F}}$ positive fuzzy numbers in $\mathbb{R}$. We also present fuzzy sets in $\mathbb{R}^n$ by $\mathcal{F}(\mathbb{R}^n)$.

For $0 < \alpha \leq 1$ and $A \in \mathbb{R}_{\mathcal{F}}$, we denote the α-cuts by $[A]_\alpha = \{x \in \mathbb{R}; A(x) \geq \alpha\}$ and $[A]_0 = \overline{\{x \in \mathbb{R}; A(x) > 0\}}$. We call $[A]_0$, the support of the fuzzy number A and denote it by supp(A).

1.2 Interactivity

The notion of interactivity was introduced by Zadeh ([23]), later Fuller, Carlsson, and Majlender associated this concept with the definition of a joint possibility distribution [5].

Definition 2. *[16] A joint possibility distribution (JPD) of n fuzzy numbers $A_1, \ldots A_n$ is a fuzzy relation $J \in \mathcal{F}(\mathbb{R}^n)$ such that, for every $y \in \mathbb{R}$*

$$A_i(y) = \bigvee_{(x_1,\ldots,x_n):x_i=y} J(x_1, \ldots, x_n). \tag{1}$$

The fuzzy relation J_t given by $J_t(x_1, ..., x_n) = A_1(x_1)tA_2(x_2)t...tA_n(x_n)$ corresponds to the joint possibility distribution of $A_1, ..., A_n \in \mathbb{R}_{\mathcal{F}}$ based on a t-norm t. When the t-norm is given by the minimum t-norm $J_\wedge$, we say that the numbers are *non-interactive*. Otherwise, they are *interactive* (resp. more precisely J*-interactive*). Interactivity is a property of variables rather than fuzzy sets. The JPD describes the simultaneously attainable states of variables, represented by fuzzy sets. Not every relation of interactivity is based on a t-norm. Carlsson et al. in [5] proposed a type of interactivity called *complete correlation* or *linear interactivity* (see a recent generalization in [13]).

Definition 3. *[5] Fuzzy numbers are said to be completely correlated, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their JPD is defined by*

$$J(x, y) = A(x)\chi_{\{qx+r=y\}}(x, y) = B(y)\chi_{\{qx+r=y\}}(x, y), \tag{2}$$

where $\chi_{\{qx+r=y\}}$ stands for the characteristic function of the line $\{(x, y) \in \mathbb{R} \mid qx + r = y\}$.

Definition 4. *[5] Fuzzy numbers A and B are said to be completely positively (negatively) correlated if q is positive (negative) in (2).*

Another generalization of complete correlation to so called F*-correlation* is given in [26], where the linear correlation is a special case of it. The authors also define arithmetic operations with F-correlated fuzzy numbers.

1.3 Sup-J Extension Principle

The sup-J extension principle is a generalization of the well-known Zadeh extension principle.

Definition 5. *[5, 14] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, \dots, A_n \in \mathbb{R}_{\mathcal{F}}$. The sup-J extension of f to a fuzzy function $f_J : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R})$ is defined by*

$$f_J(A_1, \dots, A_n)(y) = \bigvee_{y=f(x_1,\dots,x_n)} J(x_1, \dots, x_n). \tag{3}$$

If fuzzy numbers are non-interactive, then the sup-J extension principle is Zadeh's extension principle. In [6], some conditions are presented under which the sum of interactive and non-interactive fuzzy numbers coincides. Moreover, in [8], Coroianu and Fuller provided a key condition under which these two extension principles produce the same result for any continuous function. In [14], a new method is proposed to calculate the sum and differences of two fuzzy numbers. The method is based on the construction of a family of joint possibility distributions $\{J_\gamma\}$ such that the parameter γ represents the degree of interactivity. The higher the value of γ, the smaller the interactivity is. Only for $\gamma = 1$, it contains the main diagonal. The sup-J extension principle then provides the arithmetic. Moreover, we have the connection between the proposed subtraction and the Hukuhara difference, the generalized Hukuhara, and the generalized differences. Note that Coroianu [8] proved that the Zadeh and sup-J extension principle produced the same results when the joint possibility distribution under consideration coincides with $J_\wedge$ on a *main diagonal.* The next theorem provides a method for computing the α-cuts of the sup-J extension principle and can be interpreted as a generalization of Nguyen's theorem [21].

Theorem 1. *[5,14] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, \dots, A_n \in \mathbb{R}_{\mathcal{F}}$ such that $[J]_\alpha$ is non-empty and compact for all $\alpha \in [0, 1]$. Then $f_J(A_1, \dots, A_n)$ is a fuzzy number, and*

$$[f_J(A_1, \dots, A_n)]_\alpha = f([J]_\alpha), \quad \forall \alpha \in [0, 1]. \tag{4}$$

Moreover,

$$[f_J(A_1,\ldots,A_n)]_\alpha = \left[\bigwedge_{(x_1,\ldots,x_n)\in[J]_\alpha} f(x_1,\ldots,x_n), \bigvee_{(x_1,\ldots,x_n)\in[J]_\alpha} f(x_1,\ldots,x_n)\right]. \tag{5}$$

As we proceed with the experiments, we will display the outcomes of various divisions. To make it clearer, we revisit the following definition.

Definition 6. *[25] Let $A, B, C \in \mathbb{R}_\mathcal{F}$ and $A = [a^-, a^+]$, $B = [b^-, b^+]$, $C = [c^-, c^+]$. The g-division is defined as follows*

$$C = B \div_g A \iff \begin{cases} (i) & B = (A \cdot C), \\ (ii) & A = (B \cdot C^{-1}), \end{cases} \tag{6}$$

where $C^{-1} = [\frac{1}{c^+}, \frac{1}{c^-}]$.

For the purpose of our study, we limit our attention to the division of positive fuzzy numbers.

Remark 1. [19, 25] Let $A, B \in \mathbb{R}^+_\mathcal{F}$. If $B \div_g A = C \in \mathbb{R}^+_\mathcal{F}$ exists, then we have the following two cases.

Case I: If $b^-_\alpha a^+_\alpha \le b^+_\alpha a^-_\alpha$ for all $\alpha \in [0, 1]$, then:

$$c^-_\alpha = \frac{b^-_\alpha}{a^-_\alpha} \quad \text{and} \quad c^+_\alpha = \frac{b^+_\alpha}{a^+_\alpha} \tag{7}$$

Case II: If $b^-_\alpha a^+_\alpha \ge b^+_\alpha a^-_\alpha$ for all $\alpha \in [0, 1]$, then:

$$c^-_\alpha = \frac{b^+_\alpha}{a^+_\alpha} \quad \text{and} \quad c^+_\alpha = \frac{b^-_\alpha}{a^-_\alpha}. \tag{8}$$

2 Multiplication and Division of Interactive Fuzzy Numbers

The concept of interactive fuzzy numbers and their operations is specialized, and there is no universally agreed-upon standard. However, we provide a general idea of how to define a division for interactive fuzzy numbers based on the joint possibility distribution and the sup-J extension principle.

The joint possibility distribution J captures the relationship between the elements of interactive fuzzy numbers. One way to define the division between two interactive fuzzy numbers based on the joint possibility distribution is as follows.

Definition 7. *Let $A, B \in \mathbb{R}^+_\mathcal{F}$ and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of A, B. The division $\frac{B}{A}$ based on J is defined by*

$$\frac{B}{A} = \{z \in \mathbb{R} \mid \exists x \in A, \exists y \in B, J(x, y) \ge J(x, z) \cdot J(z, y)\}. \tag{9}$$

The condition $J(x,y) \geq J(x,z) \cdot J(z,y)$ essentially says that the joint possibility of x and y together should be greater than or equal to the product of the joint possibilities $J(x,z)$ and $J(z,y)$. In other words, it ensures that the joint possibility distribution supports the idea that z is a valid result of dividing y by x. So, the set $\frac{B}{A}$ includes elements z for which there exist corresponding elements x in A and y in B such that the joint possibility distribution J is consistent with the division $z = \frac{y}{x}$.

Based on the sup-J extension principle, the interactive product of fuzzy numbers can be defined as follows.

Definition 8. *[7] Let $A, B \in \mathbb{R}_{\mathcal{F}}$ and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of A, B. The multiplication $A \odot_J B$ with respect to J is defined by*

$$(A \odot_J B)(z) = \bigvee_{x \cdot y = z} J(x,y), \quad z \in \mathbb{R}. \tag{10}$$

Similarly, we define the interactive division of two positive fuzzy numbers A and B with respect to their joint possibility distribution J based on the sup-J extension principle.

Definition 9. *Let $A, B \in \mathbb{R}^+_{\mathcal{F}}$ and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of A, B. The division $B \div_J A$ with respect to J (J-division) is defined by*

$$(B \div_J A)(z) = \bigvee_{y/x = z} J(x,y), \quad z \in \mathbb{R}. \tag{11}$$

To define alpha cuts for interactive multiplication and division of two fuzzy numbers based on a joint possibility distribution, we use the standard definition of alpha cuts of a fuzzy number and Theorem 1.

The α-cut of a fuzzy number obtained as the interactive product of two fuzzy numbers that are interactive with respect to a joint possibility distribution is defined as follows.

Definition 10. *Let $A, B \in \mathbb{R}_{\mathcal{F}}$ and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of A, B. The α-cut of multiplication $A \odot_J B$ is defined by*

$$[A \odot_J B]_\alpha = \{z \in \mathbb{R} \mid \exists x \in A, y \in B : xy = z, \ J(x,y) \geq \alpha\}, \quad \alpha \in [0,1]. \tag{12}$$

We use Theorem 1 and assume a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_\odot(x,y) = xy$. Then, we obtain

$$[A \odot_J B]_\alpha = f_\odot([J(A,B)]_\alpha) = f_\odot(\big\{(x,y) \in \mathbb{R}^2 \mid x \in A, y \in B : J(x,y) \geq \alpha\big\}) \tag{13}$$

$$= \left[\bigwedge_{(x,y) \in [J(A,B)]_\alpha} x \cdot y, \bigvee_{(x,y) \in [J(A,B)]_\alpha} x \cdot y \right], \quad \alpha \in [0,1]. \tag{14}$$

In the same way, the α-cut of an interactive division (J-division) of two fuzzy numbers based on a joint possibility distribution is defined as follows.

Definition 11. *Let $A, B \in \mathbb{R}^+_{\mathcal{F}}$ and let $J \in \mathcal{F}(\mathbb{R}^2)$ be a joint possibility distribution of A, B. The α-cut of the J-division $B \div_J A$ is defined by*

$$[B \div_J A]_\alpha = \left\{ z \in \mathbb{R} \mid \exists x \in A, y \in B : \frac{y}{x} = z,\ J(x,y) \geq \alpha \right\}, \quad \alpha \in [0,1]. \tag{15}$$

After applying Theorem 1 and assuming a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f_\div(x,y) = \frac{y}{x}$, we obtain

$$[B \div_J A]_\alpha = f_\div([J(A,B)]_\alpha) = f_\div(\{(x,y) \in \mathbb{R}^2 \mid x \in A, y \in B : J(x,y) \geq \alpha\}) \tag{16}$$

$$= \left[\bigwedge_{(x,y)\in[J(A,B)]_\alpha} \frac{y}{x}, \bigvee_{(x,y)\in[J(A,B)]_\alpha} \frac{y}{x} \right], \quad \alpha \in [0,1]. \tag{17}$$

3 *J*-division of Completely Correlated Fuzzy Numbers

In this section, we investigate the J-division where the joint possibility distribution J is specified by linear interactivity, i.e., the fuzzy numbers are completely correlated (see Definition 3), we denote it J_c. Let us restrict ourselves to positively correlated fuzzy numbers (see Definition 4).

Let $A, B \in \mathbb{R}^+_{\mathcal{F}}$ and let $J_c \in \mathcal{F}(\mathbb{R}^2)$ be the joint possibility distribution of A, B defined by (2). For the α-cuts of J_c we have

$$[J_c]_\alpha = \{(x, qx + r) \in \mathbb{R}^2 \mid x = (1-t)a^-_\alpha + t a^+_\alpha, t \in [0,1]\}, \tag{18}$$

where

$$[A]_\alpha = [a^-_\alpha, a^+_\alpha] \quad \text{and} \quad [B]_\alpha = q[A]_\alpha + r; \quad q, r \in \mathbb{R}, q > 0, \tag{19}$$

for any $\alpha \in [0,1]$.

Then for the J_c-division with J_c defined by (2), we obtain

$$\begin{aligned}[B \div_{J_c} A]_\alpha &= cl \left\{ \frac{y}{x} \in \mathbb{R} \mid A(x) > \alpha, qx + r = y \right\} \\ &= cl \left\{ q + r\frac{1}{x} \in \mathbb{R} \mid A(x) > \alpha \right\} \\ &= q + r \cdot cl \left\{ \frac{1}{x} \in \mathbb{R} \mid A(x) > \alpha \right\},\end{aligned}$$

that is for all $q, r \in \mathbb{R}$, $q > 0$, and $\alpha \in [0,1]$,

$$[B \div_{J_c} A]_\alpha = \begin{cases} q + r\left[\frac{1}{a^-_\alpha}, \frac{1}{a^+_\alpha}\right], & r < 0 \\ q + r\left[\frac{1}{a^+_\alpha}, \frac{1}{a^-_\alpha}\right], & r \geq 0. \end{cases} \tag{20}$$

If A and B are completely correlated with $r = 0$, that is, $[B]_\alpha = q[A]_\alpha$, $q > 0$, for all $\alpha \in [0, 1]$. From (20), we have

$$[B \div_{J_c} A]_\alpha = q + 0 \times [A]_\alpha^{-1} = \{q\},$$

that is $B \div_{J_c} A$ is a fuzzy point.

3.1 Comparison of J_c-division with g-division and Zadeh's division

One of our objectives is to find alternatives or competitors for current divisions such as g-division (generalized Hukuhara division) and division based on Zadeh's extension principle. In this section, we compare the division mentioned for completely positively correlated fuzzy numbers. Now, let us consider g-division (Definition 6) for two completely positively correlated fuzzy numbers. Let $A, B \in \mathbb{R}_{\mathcal{F}}^+$ be completely correlated (Definition 3) and let $B \div_g A$ exist. We investigate the conditions for Case I and Case II in Remark 1. Based on (19), we obtain the following.

(i) For all $q, r \in \mathbb{R}$, such that $q > 0$ and $r < 0$,

$$\begin{aligned} b_\alpha^- a_\alpha^+ &\leq b_\alpha^+ a_\alpha^- \\ r a_\alpha^+ &\leq r a_\alpha^-, \end{aligned}$$

i.e., Case I holds for all $\alpha \in [0, 1]$. Moreover based on (19) and (20), for $r < 0$,

$$[B \div_{J_c} A]_\alpha = \left[q + \frac{r}{a_\alpha^-}, q + \frac{r}{a_\alpha^+}\right] = \left[\frac{b_\alpha^-}{a_\alpha^-}, \frac{b_\alpha^+}{a_\alpha^+}\right] = [B \div_g A]_\alpha \quad \text{(Case I)},$$

(ii) For all $q, r \in \mathbb{R}$, such that $q > 0$ and $r \geq 0$

$$\begin{aligned} b_\alpha^- a_\alpha^+ &\geq b_\alpha^+ a_\alpha^- \\ r a_\alpha^+ &\geq r a_\alpha^-, \end{aligned}$$

i.e., Case II holds for all $\alpha \in [0, 1]$. Moreover based on (19) and (20), for $r \geq 0$,

$$[B \div_{J_c} A]_\alpha = \left[q + \frac{r}{a_\alpha^+}, q + \frac{r}{a_\alpha^-}\right] = \left[\frac{b_\alpha^+}{a_\alpha^+}, \frac{b_\alpha^-}{a_\alpha^-}\right] = [B \div_g A]_\alpha \quad \text{(Case II)}.$$

which yileds to

$$[B \div_g A]_\alpha = [B \div_{J_c} A]_\alpha.$$

If we consider the division $B \div_z A$ based on Zadeh's extension principle of two completely positively correlated fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}}^+$, we obtain

$$[B \div_z A]_\alpha = \left[\frac{q a_\alpha^- + r}{a_\alpha^+}, \frac{q a_\alpha^+ + r}{a_\alpha^-}\right]$$

for $q, r \in \mathbb{R}$, $q > 0$ and $\alpha \in [0, 1]$. Therefore,

$$[B \div_{J_c} A]_\alpha \subset [B \div_z A]_\alpha$$

for all $\alpha \in [0, 1]$.

The comparisons show that for positive fuzzy numbers that are completely positively correlated, the J_c-division coincides with the g-division and has narrower range with respect to the Zadeh's division.

4 Examples

In this section, we compare the J_c-division based on linear interactivity (complete correlation) with the g-division and division based on Zadeh's extension principle in particular examples with respect to different parameters q and r. Our objective is to apply division operations within difference equations that have application in the analysis of real-world phenomena, including finance problems, time series analysis, and population models [9, 22].

Example 1. Consider $A = (1, 2, 3)$ and $B = (4, 6, 8)$ as triangular positive fuzzy numbers that are completely correlated with parameters $q = 2$, $r = 2$, with their α-cuts defined by $[A]_\alpha = [1 + \alpha, 3 - \alpha]$ and $[B]_\alpha = [4 + 2\alpha, 8 - 2\alpha]$. Then the α-cuts for the J_c-division $[B \div_{J_c} A]_\alpha$ are depicted in Fig. 1 and the α-cuts for the g-division (Case II) $[B \div_g A]_\alpha$ are illustrated in Fig. 2. For comparison, we show in both images the α-cuts for the Zadeh's division $[B \div_z A]_\alpha$.

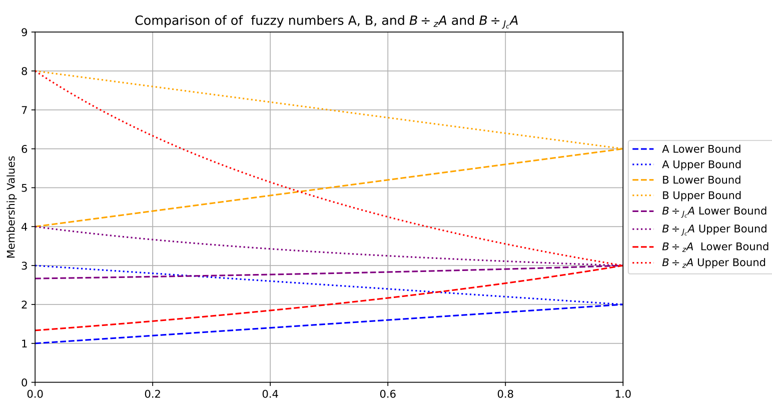

Fig. 1. α-cuts for J_c-division $B \div_{J_c} A$ and Zadeh's division $B \div_z A$(Example 1).

Example 2. Let $A = (2, 3, 4)$ and $B = (2, 4, 6)$ be triangular positive fuzzy numbers that are completely correlated with parameters $q = 2$, $r = -2$, and the α

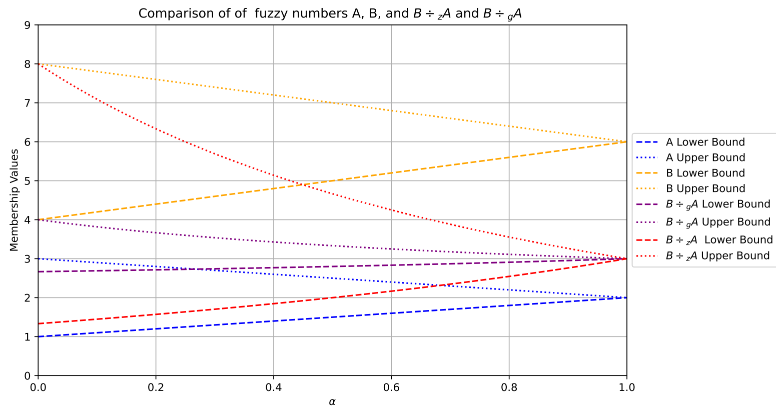

Fig. 2. α-cuts for g-division $B \div_g A$ and Zadeh's division $B \div_z A$(Example 1).

cuts are given by $[A]_\alpha = [2 + \alpha, 4 - \alpha]$ and $[B]_\alpha = [2 + 2\alpha, 6 - 2\alpha]$. Then the α-cuts for the J_c-division $[B \div_{J_c} A]_\alpha$ are shown in Fig. 3 and the α-cuts for the g-division (Case I) $[B \div_g A]_\alpha$ are shown in Fig. 4. Both divisions mentioned are in comparison with the Zadeh's division.

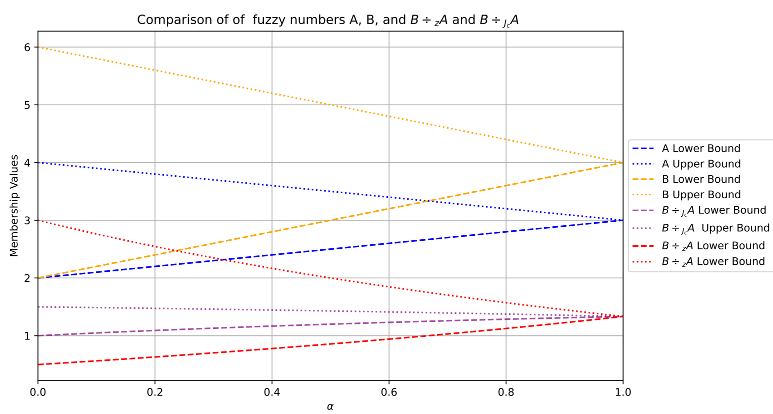

Fig. 3. α-cuts for J_c-division $B \div_{J_c} A$ and Zadeh's division $B \div_z A$ (Example 2).

The results of both examples align perfectly with our theoretical discussion, indicating that the g-division agrees with the J_c-division for positive fuzzy numbers. A comparison of the ambiguity of Zadeh's division with other division methods exposes a broader uncertainty spectrum linked to Zadeh's division.

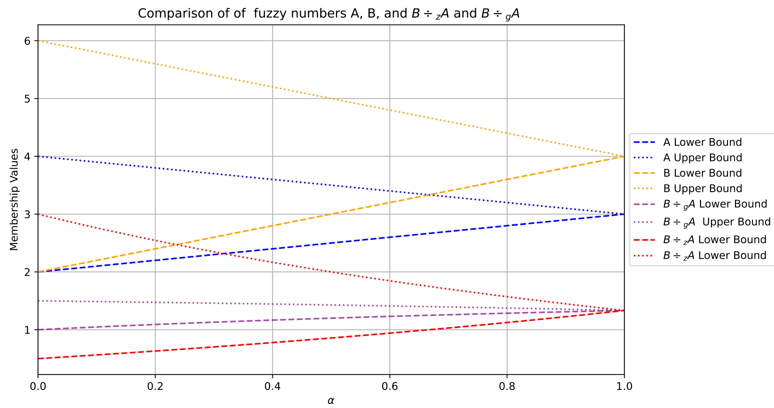

Fig. 4. α-cuts for g-division $B \div_g A$ and Zadeh's division $B \div_z A$ (Example 2).

5 Conclusion

Our study demonstrates that our proposed division method yields fuzzy numbers with narrower widths compared to other methods, supported by joint possibility distributions. Furthermore, we establish connections between our method and existing techniques such as the generalized Hukuhara division and Zadeh's extension principle. We provide illustrative examples to highlight the practical advantages of our approach. Future research on J-division encompassing all cases of positive and negative fuzzy numbers holds promise for advancing fuzzy logic's ability to manage uncertainty and interactivity.

References

1. Ahmad, M.Z., Baets, B.D.: A predator-prey model with fuzzy initial populations. In: IFSA/EUSFLAT Conf. (2009), https://api.semanticscholar.org/CorpusID:15834610
2. Baetens, Jan and De Baets, Bernard: Incorporating fuzziness in spatial susceptible-infected epidemic models. In: Proceedings of the joint 2009 International Fuzzy Systems Association world congress and 2009 European Society for Fuzzy Logic and Technology conference. pp. 201–206 (2009)
3. Bede, B.: Fuzzy Analysis, pp. 137–170. Springer Berlin Heidelberg (2013). https://doi.org/10.1007/978-3-642-35221-8_8
4. Cabral, V., Barros, L.: Fuzzy differential equation with completely correlated parameters. Fuzzy Sets and Systems **265**, 86–98 (2015). https://doi.org/https://doi.org/10.1016/j.fss.2014.08.007, theme: Analysis
5. Carlsson, C., Fullér, R., et al.: Additions of completely correlated fuzzy numbers. In: 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No. 04CH37542). vol. 1, pp. 535–539. IEEE (2004)
6. Coroianu, L.: Necessary and sufficient conditions for the equality of the interactive and non-interactive sums of two fuzzy numbers. Fuzzy Sets and Systems **283**, 40–55 (2016)

7. Coroianu, L., Fullér, R.: On multiplication of interactive fuzzy numbers. In: 2013 IEEE 11th International Symposium on Intelligent Systems and Informatics (SISY). pp. 181–185. IEEE (2013)
8. Coroianu, L., Fullér, R.: Necessary and sufficient conditions for the equality of interactive and non-interactive extensions of continuous functions. Fuzzy Sets and Systems **331**, 116–130 (2018)
9. Deeba, E., De Korvin, A.: Analysis by fuzzy difference equations of a model of co2 level in the blood. Applied mathematics letters **12**(3), 33–40 (1999)
10. Diamond, P., Kloeden, P.: Metric spaces of fuzzy sets. Fuzzy Sets and Systems **100**, 63–71 (1999)
11. Dubois, D., Prade, H.: Possibility Theory: An Approach to Computerized Processing of Uncertainty. Springer US (1988), https://books.google.cz/books?id=S1N3QgAACAAJ
12. Dubois, D., Prade, H.: Additions of interactive fuzzy numbers. IEEE Transactions on Automatic Control **26**(4), 926–936 (1981)
13. Esmi, E., Sánchez, D.E., Wasques, V.F., de Barros, L.C.: Solutions of higher order linear fuzzy differential equations with interactive fuzzy values. Fuzzy Sets and Systems **419**, 122–140 (2021)
14. Esmi, E., Wasques, V.F., de Barros, L.C.: Addition and subtraction of interactive fuzzy numbers via family of joint possibility distributions. Fuzzy Sets and Systems **424**, 105–131 (2021)
15. Full, R.: er, t. keresztfalvi, t-norm-based addition of fuzzy intervals. Fuzzy Sets and Systems **51**, 155–159 (1992)
16. Fullér, R., Majlender, P.: On interactive fuzzy numbers. Fuzzy Sets and Systems **143**(3), 355–369 (2004). https://doi.org/https://doi.org/10.1016/S0165-0114(03)00180-5
17. Hong, D.H.: On shape-preserving additions of fuzzy intervals. Journal of Mathematical Analysis and applications **267**(1), 369–376 (2002)
18. Kawaguchi, M.F., Da-te, T.: Some algebraic properties of weakly non-interactive fuzzy numbers. Fuzzy Sets and Systems **68**(3), 281–291 (1994)
19. Khastan, A., Alijani, Z.: On the new solutions to the fuzzy difference equation xn+ 1= a+ bxn. Fuzzy Sets and Systems **358**, 64–83 (2019)
20. Lakshmikantham, V., Trigiante, V.: Theory of difference equations numerical methods and applications, vol. 251. CRC Press (2002)
21. Nguyen, H.T.: A note on the extension principle for fuzzy sets. Journal of Mathematical Analysis and Applications **64**(2), 369–380 (1978). https://doi.org/https://doi.org/10.1016/0022-247X(78)90045-8
22. Papaschinopoulos, G., Papadopoulos, B.K.: On the fuzzy difference equation x n+ 1= a+ b/xn. Soft Computing **6**, 456–461 (2002)
23. Scheerlinck, K., Vernieuwe, H., De Baets, B.: Zadeh's extension principle for continuous functions of non-interactive variables: A parallel optimization approach. IEEE Transactions on Fuzzy Systems **20**(1), 96–108 (2011)
24. Snášel, V., Perfilieva, I., Singh, M., Pant, M., Alijani, Z., Kong, L.: A rankability-based fuzzy decision making procedure for oil supplier selection. Applied Soft Computing **149**, 110956 (2023)
25. Stefanini, L.: A generalization of hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy sets and systems **161**(11), 1564–1584 (2010)
26. Wasques, V., Laiate, B., Santo Pedro, F., Esmi, E., de Barros, L.C.: Interactive fuzzy fractional differential equation: Application on hiv dynamics. In: Information Processing and Management of Uncertainty in Knowledge-Based Systems. pp. 198–211. Springer International Publishing, Cham (2020)