Axiomatizing the Logic of Ordinary Discourse*

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Abstract. Most non-classical logics are subclassical, that is, every inference/theorem they validate is also valid classically. A notable exception is the three-valued propositional *Logic of Ordinary Discourse* (OL) proposed and extensively motivated by W.S. Cooper as a more adequate candidate for formalizing everyday reasoning (in English). OL challenges classical logic not only by rejecting some theses, but also by accepting non-classically valid principles, such as so-called Aristotle's and Boethius' theses. Formally, OL shows a number of unusual features – it is non-structural, connexive, paraconsistent and contradictory – making it all the more interesting for the mathematical logician. We present our recent findings on OL and its structural companion (that we call sOL). We introduce Hilbert-style multiple-conclusion calculi for OL and sOL that are both modular and analytic, and easily allow us to obtain single-conclusion axiomatizations. We prove that sOL is algebraizable and single out its equivalent semantics, which turns out to be a discriminator variety generated by a three-element algebra. Having observed that sOL can express the connectives of other three-valued logics, we prove that it is definitionally equivalent to an expansion of the three-valued logic $\mathcal{J}3$ of D'Ottaviano and da Costa, itself an axiomatic extension of paraconsistent Nelson logic.

Keywords: Ordinary discourse · Multiple-conclusion systems · Algebraic semantics · Connexive logics.

1 Introduction

Most non-classical propositional systems result from weakening Boolean two-valued logic one way or another: this applies in general to logics in the fuzzy, many-valued, relevance and substructural families. The majority of three-valued logics are also subclassical in this sense (see [9] for an overview). A remarkable exception is represented by the *Logic of Ordinary Discourse* introduced by W.S. Cooper [5], a propositional system (henceforth denoted OL) remarkable in other ways as well.

As the name indicates, OL was proposed as a logic for formalizing *ordinary discourse*, that is, it was designed to model the everyday usage of natural language connectives (in particular the *if-then*). In this respect, the main point of departure

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of OL from classical logic concerns those conditional sentences having a false antecedent. Following a famous suggestion of Quine, Cooper regards such sentences as lacking a truth value, but formally represents this condition by employing a third (or "gap") value $\mathbf{1/2}$ besides the classical $\mathbf{1}$ and $\mathbf{0}$. OL may thus be defined by means of valuations over a three-element logical matrix (Figure 1) over the set of truth values $\{\mathbf{0}, \mathbf{1}, \mathbf{1/2}\}$ where, interestingly, both $\mathbf{1}$ and the gap value $\mathbf{1/2}$ are designated. However, as we shall soon discuss, OL is *not* determined by this three-valued matrix in the usual sense, for in its definition not all valuations are allowed.

$\land$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$\lor$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$

$\to$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1/2}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$

	$\neg$
$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$

Fig. 1. Truth tables for OL [5, Sec. 5].

OL validates certain formulas that are not classical tautologies – notably so-called *Boethius' thesis* $(\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$ and *Aristotle's thesis* $\neg(\neg\psi \to \psi)$ – suggesting that it may be viewed as a *connexive* logic [14]. On the other hand, some classical tautologies (such as the explosion principle $\varphi \to (\neg\varphi \to \psi)$) are not unconditionally valid in OL, making it incomparable with (not weaker than) classical propositional logic.

Cooper [5] dicusses several examples supporting the claim that OL provides a better formal model of ordinary discourse than classical logic. Here we may mention but one, a puzzle that can be traced back to Aristotle, as reconstructed by J. Łukasiewicz [5, pp. 312-3]. In Aristotle's view, no proposition can imply its own negation, hence $\neg(\neg\psi \to \psi)$ should be a tautology. But, classically, $(\varphi \to \psi) \land (\neg\varphi \to \psi) \vdash \neg\psi \to \psi$. Hence, by contraposition, $(\varphi \to \psi) \land (\neg\varphi \to \psi)$ should be contradictory, which seems counter-intuitive. Now, in OL, Aristotle's thesis $\neg(\neg\psi \to \psi)$ is indeed a tautology. We also have, as in classical logic, that $(\varphi \to \psi) \land (\neg\varphi \to \psi) \vdash \neg\psi \to \psi$ and $(\varphi \to \psi) \land (\neg\varphi \to \psi)$ is satisfiable. OL, however, rejects the contraposition rule ($\alpha \vdash \beta$ does not entail $\neg\beta \vdash \neg\alpha$), so $\neg(\neg\psi \to \psi) \nvdash \neg((\varphi \to \psi) \land (\neg\varphi \to \psi))$, avoiding the counter-intuitive consequence. We shall not further discuss the adequacy of OL with respect to the proposed applications (refer to [5]), but we wish to draw attention to some of its unusual features.

For one thing, OL is not even a logic in the Tarskian sense, for its consequence relation is *non-structural*, i.e. not closed under uniform substitution. This is witnessed, for instance, by the explosion principle, which OL endorses for atomic formulas ($p, \neg p \vdash q$) but not for arbitrary ones ($\varphi, \neg\varphi \nvdash \psi$). In consequence, OL is not characterized by truth tables in the usual way; it can, however, be semantically characterized by the three-valued matrix described in Figure 1 if we require valuations to assign only classical values ($\mathbf{0}$ or $\mathbf{1}$) to the propositional variables.

Let us mention a few other unusual features of OL, whose language consists of a conjunction ($\land$), a disjunction ($\lor$), an implication ($\to$) and a negation ($\neg$). The usual De Morgan laws relating conjunction and disjunction via negation hold, and indeed either $\land$ or $\lor$ could be omitted from the primitive signature. However, neither the distributive nor the absorption laws between conjunction and disjunction hold. Indeed, $\land$ and $\lor$ do not give rise to a lattice structure in the expected way (cf. Proposition 1 below); for this reason they have been called *quasi-conjunction* and *quasi-disjunction* in the literature (see [6, Sec. 6.3] for a discussion of their motivation). A novel contribution to an informal reading of these connectives may perhaps be based on our algebraic analysis of OL. As we shall see (Section 4), OL may be viewed as an expansion of Da Costa and D'Ottaviano's logic $\mathcal{J}3$, which is in turn an axiomatic extension of paraconsistent Nelson logic. These are substructural logics based on residuated structures having both a lattice (or *additive*) conjunction and a *multiplicative* conjunction; on the algebraic models, the latter is realized by a monoid operation having (in our notation) $\mathbf{1/2}$ as neutral element. This suggests that Cooper's connective $\wedge$ is perhaps best thought of as a multiplicative conjunction having $\vee$ as its De Morgan dual (defined by $x \vee y := \neg(\neg x \wedge \neg y)$), the truncated sum of Łukasiewicz logic being another example of this kind of disjunction.

The behaviour of $\vee$ in OL is indeed peculiar, even in isolation. In fact, the preceding observations entail that disjunction introduction is not a sound rule (in general, we have $\varphi \nvdash \varphi \vee \psi$). In many logics, another key feature of the disjunction is that the truth value of a formula $\varphi \vee \psi$ is designated if and only if either φ or ψ is assigned a designated value. This does not hold in OL (one has, for instance $\mathbf{1/2} \vee \mathbf{0} = \mathbf{0}$). In consequence, a classical tautology such as $\varphi \vee (\varphi \to \psi)$ is contigent in OL, although every valuation assigns a designated value either to φ or to $\varphi \to \psi$. The behaviour of $\to$ within OL appears to be more standard, at least in isolation. Unlike the disjunction, the implication is not definable from the other connectives in any of the usual ways. Indeed, it is easy to see that $\to$ is not definable at all in the language $\{\wedge, \vee, \neg\}$, for the set $\{\mathbf{0}, \mathbf{1}\}$ is closed under all these operations but not under the implication. On the other hand, the truth constants (all three of them) are definable. For instance, one may let $\mathbf{1/2} := \varphi \to (\neg \varphi \to \varphi)$, $\mathbf{1} := p \vee \neg p$ and $\mathbf{0} := \neg \mathbf{1}$ (here p needs to be atomic, while φ is arbitrary).

Lastly, observe that OL is not only a paraconsistent (for $\varphi, \neg \varphi \nvdash \psi$) but actually a *contradictory* logic in H. Wansing's sense [15], that is, there exists a formula (e.g. $\mathbf{1/2}$, defined as above) such that both $\vdash \mathbf{1/2}$ and $\vdash \neg \mathbf{1/2}$ hold. This has the interesting consequence that OL does not admit any non-trivial structural extension: if we were to close the consequence relation of OL under uniform substitutions, then any formula φ would be valid, being (by the explosion rule, now applicable to arbitrary formulas) a consequence of the set of valid formulas $\{\mathbf{1/2}, \neg \mathbf{1/2}\}$.

We introduce multiple- and single-conclusion Hilbert-style axiomatizations for OL and for its structural companion sOL (Sections 3), thus filling the gap in the literature concerning a standard axiomatization of OL. Our calculi are modular (i.e., obtained by joining independent calculi over smaller signatures), easily allowing us to characterize a number of fragments of OL/sOL. In the multiple-conclusion setting, they are also analytic. We obtain these axiomatizations via the methods developed in [13,3]. In Section 4 we give an alternative axiomatization for sOL and we prove that sOL is algebraizable with respect to the quasi-variety $\mathbb{OL}$ (coinciding with the variety) generated by the algebra $\mathbf{O}_3$, the algebraic reduct of Cooper's three-valued matrix. $\mathbb{OL}$ turns out to be a discriminator variety (Theorem 8), making sOL a nearly functionally complete logic (adding to its language either the constant $\mathbf{1}$ or $\mathbf{0}$ makes it functionally complete). Indeed, we show that sOL is definitionally equivalent to an expansion of Da Costa and D'Ottaviano's three-valued logic $\mathcal{J}3$, which is in turn an axiomatic extension of paraconsistent Nelson logic. The final Section 5 discusses potential future developments, notably a more extensive study of the algebraic counterpart of sOL (especially in connection with the other above-mentioned non-classical logics), and the extension of the present approach to other logics definable from the truth tables of OL (e.g., the algebraizable fragments of sOL, logics resulting from alternative choices of designated elements or those determined by the definable orders of $\mathbf{O}_3$).

2 Logical preliminaries

A *propositional signature* Σ is a collection of symbols called *connectives*; to each of them is assigned a natural number called *arity*. Given a countable set P of propositional variables, the *language over Σ generated by P* is the absolutely free algebra over Σ freely generated by P, denoted $\mathbf{L}_\Sigma(P)$. Its carrier is denoted by $L_\Sigma(P)$ and its elements are called *formulas*. The collection of all subformulas of a formula φ is denoted by $\mathsf{subf}(\varphi)$. Similarly, the set of all propositional variables occurring in $\varphi \in L_\Sigma(P)$ is denoted by $\mathsf{props}(\varphi)$.

A *single-conclusion logic (over Σ)* is a (non-necessarily structural) consequence relation $\vdash$ on $L_\Sigma(P)$ and a *multiple-conclusion logic (over Σ)* is a generalized consequence relation $\rhd$ on $L_\Sigma(P)$ — see [10, Secs. 1.12, 1.16] for precise definitions. The *single-conclusion companion* of a given multiple-conclusion logic $\rhd$ is the single-conclusion logic $\vdash_\rhd$ such that $\Phi \vdash_\rhd \psi$ if, and only if, $\Phi \rhd \{\psi\}$. We adopt the convention of omitting curly braces when writing sets of formulas in statements involving (generalized) consequence relations. The complement of a given $\rhd$, i.e., $\mathcal{P}(L_\Sigma(P)) \times \mathcal{P}(L_\Sigma(P)) \setminus \rhd$, will be denoted by $\blacktriangleright$.

Given a signature Σ, a *three-valued matrix over Σ* (or *Σ-matrix*) is a structure $\mathbb{M} := \langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is an algebra over Σ with carrier $O_3 := \{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}$, and $D \subseteq O_3$ (the set of *designated values*). The algebra $\mathbf{A}$ assigns to each k-ary connective in Σ a k-ary operation $©_\mathbf{A} : O_3^k \to O_3$, called the *interpretation* of © in $\mathbb{M}$. For each formula $\varphi(p_1, \ldots, p_k)$ on k propositional variables, we denote by $\varphi_\mathbf{A}$ the k-ary derived operation on $\mathbf{A}$ induced by φ in the standard way. Homomorphisms from $\mathbf{L}_\Sigma(P)$ to $\mathbf{A}$ are called $\mathbb{M}$-*valuations*. Every $\mathbb{M}$ determines multiple-conclusion and single-conclusion substitution-invariant consequence relations in the following way:

$$\Phi \rhd_\mathbb{M} \Psi \text{ iff, for no } \mathbb{M}\text{-valuation } v,\ v[\Phi] \subseteq D \text{ and } v[\Psi] \subseteq O_3 \setminus D$$
$$\Phi \vdash_\mathbb{M} \psi \text{ iff, for no } \mathbb{M}\text{-valuation } v,\ v[\Phi] \subseteq D \text{ and } v(\psi) \in O_3 \setminus D$$

We may also consider proper subsets of $\mathbb{M}$-valuations, leading to potentially non-structural consequence relations, as Cooper did for OL. Following [5], we consider here the particular class that we call *b$\mathbb{M}$-valuations*, comprising those valuations that assign to propositional variables either $\mathbf{0}$ or $\mathbf{1}$ (complex formulas may still be assigned the value $\mathbf{1/2}$ even under this restriction, depending on the interpretations of the connectives). We thus obtain the following logics:

$$\Phi \rhd^{\mathsf{biv}}_\mathbb{M} \Psi \text{ iff, for no b}\mathbb{M}\text{-valuation } v,\ v[\Phi] \subseteq D \text{ and } v[\Psi] \subseteq O_3 \setminus D$$
$$\Phi \vdash^{\mathsf{biv}}_\mathbb{M} \psi \text{ iff, for no b}\mathbb{M}\text{-valuation } v,\ v[\Phi] \subseteq D \text{ and } v(\psi) \in O_3 \setminus D$$

We use the above notions to precisely define what logics we are interested in here. Let $\Sigma_{\mathrm{OL}} := \{\neg, \vee, \wedge, \to\}$ and Σ be a signature such that $\Sigma \cap \Sigma_{\mathrm{OL}} \neq \varnothing$. We say that a three-valued Σ-matrix $\mathbb{M}$ is an OL-*matrix* when the connectives in $\Sigma \cap \Sigma_{\mathrm{OL}}$ are interpreted as in Figure 1 and the set of designated values is $\{\mathbf{1/2}, \mathbf{1}\}$. If $\Sigma = \Sigma_{\mathrm{OL}}$, $\rhd_{\mathrm{sOL}} := \rhd_\mathbb{M}$ and $\vdash_{\mathrm{sOL}} := \vdash_\mathbb{M}$ are what we call (multiple-conclusion and single-conclusion, respectively) sOL; similarly, $\rhd^{\mathsf{biv}}_{\mathrm{OL}} := \rhd^{\mathsf{biv}}_\mathbb{M}$ and $\vdash^{\mathsf{biv}}_{\mathrm{OL}} := \vdash^{\mathsf{biv}}_\mathbb{M}$ go by the name of (multiple-conclusion and single-conclusion, respectively) OL. If $\Sigma \neq \Sigma_{\mathrm{OL}}$ instead, $\rhd_\mathbb{M}$ and $\vdash_\mathbb{M}$ are called a (multiple-conclusion and single-conclusion, respectively) *fragment* of sOL if $\Sigma \subseteq \Sigma_{\mathrm{OL}}$ and an *expansion* (of a fragment of) sOL in case $\Sigma \setminus \Sigma_{\mathrm{OL}} \neq \varnothing$. In case Σ is arbitrary but $\mathbb{M}$ interprets its connectives by functions that are term-definable from the interpretations of Σ_{OL} in Figure 1, we say more generally that $\rhd_\mathbb{M}$ and $\vdash_\mathbb{M}$ are *term-definable fragments* of sOL. The same terminologies for OL are defined analogously. We will be particularly interested in the following term-definable connectives (see Figure 2 for their truth tables):

$$\Diamond x := \neg x \to x \qquad x \sqcup y := \neg \Diamond \neg (x \vee y) \wedge ((x \to y) \to \Diamond y)$$
$$x \Rightarrow y := \neg x \vee y \qquad x \sqcap y := \neg(\neg x \sqcup \neg y)$$
$$x \supset y := \Diamond x \Rightarrow y \qquad x \curlyvee y := (x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow x)$$

$\sqcap$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$1/2$	$1/2$	$1/2$	$\mathbf{0}$
$\mathbf{1}$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$\sqcup$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$1/2$	$1/2$	$\mathbf{1}$	$1/2$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{0}$	$1/2$	$\mathbf{1}$	$\mathbf{0}$

	$\diamond$
$1/2$	$1/2$
$\mathbf{1}$	$1/2$
$\mathbf{0}$	$\mathbf{0}$

$\supset$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$1/2$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\Rightarrow$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$1/2$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\curlyvee$	$1/2$	$\mathbf{1}$	$\mathbf{0}$
$1/2$	$1/2$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$

Fig. 2. Truth tables of term-definable connectives of OL/sOL.

Now we present definitions related to Hilbert-style calculi. A *multiple-conclusion Hilbert calculus* over Σ is traditionally defined as a collection R of *multiple-conclusion rules* of the form $\frac{\Phi}{\Psi}$, where $\Phi, \Psi \subseteq L_\Sigma(P)$. Here we also consider rules denoted by $\frac{\Phi}{\Psi}[\Pi]$ for $\Pi \subseteq \mathsf{props}(\Phi \cup \Psi)$, to be able to axiomatize non-substitution-invariant logics, as we shall soon explain. We will often identify $\frac{\Phi}{\Psi}$ with $\frac{\Phi}{\Psi}[\varnothing]$. Rules with $\Pi \neq \varnothing$ are referred to as *identity-instance rules*. When writing rules, we usually omit curly braces from the set notation. A *proof* of (Γ, Π) in R is a finite directed rooted tree where each node is labelled either with a set of formulas or with the discontinuation symbol $\star$, such that (i) the root is labelled with a superset of Γ; (ii) every leaf is labelled either with a set having non-empty intersection with Π or with $\star$; (iii) every non-leaf node has children determined by a substitution instance of a rule of inference of R, in the way we now detail. A rule instance r^σ applies to a node n when the antecedent of r^σ is contained in the label of n; the application results in n having exactly one child node for each formula ψ in the succedent of r^σ, which is, in turn, labelled with the same formulas as those of (the label of) n plus ψ. In case r^σ has an empty succedent, then n has a single child node labelled with $\star$. If r has the form $\frac{\Phi}{\Psi}$, then any substitution σ can be applied to instantiate it; if it is of the form $\frac{\Phi}{\Psi}[\Pi]$, however, only substitutions σ with $\sigma(p) = p$ for each $p \in \Pi$ may be applied. When we display proof trees, it is common to write as labels only the formulas introduced by the rule application, instead of the whole accumulated set of formulas. Examples of multiple-conclusion proofs may be found in Figure 4.

We write $\Gamma \rhd_{\mathsf{R}} \Pi$ whenever there is a proof of (Γ, Π) in R, and write $\vdash_{\mathsf{R}}$ for the single-conclusion companion of $\rhd_{\mathsf{R}}$. These relations are respectively multiple-conclusion and single-conclusion consequence relations that are substitution-invariant when no identity-instance rule is present in R. Single-conclusion Hilbert calculi are just the traditional Hilbert calculi, which can be seen as multiple-conclusion calculi in which only rules of the form $\frac{\Phi}{\psi}$ and $\frac{\Phi}{\psi}[\Pi]$ are allowed, where $\Phi \subseteq L_\Sigma(P)$ and $\psi \in L_\Sigma(P)$. Derivations then can be seen as linear trees, usually displayed simply as sequences of formulas. We say that a Hilbert calculus R (multiple- or single-conclusion) *axiomatizes* $\rhd$ if $\rhd = \rhd_{\mathsf{R}}$. It axiomatizes $\vdash$ in case $\vdash = \vdash_{\mathsf{R}}$.

3 Hilbert-style axiomatizations for sOL and OL

We present now, in a modular way, multiple- and single-conclusion Hilbert-style axiomatizations for sOL and OL and for fragments/expansions in which $\neg$ is present.

3.1 Multiple-conclusion

Every logic determined by a finite matrix is finitely axiomatized by a multiple-conclusion calculus [13]. Moreover, if the matrix is *monadic*, the calculus satisfies a generalized form of analyticity and can be effectively generated from the matrix description [3]. We now describe in more detail these notions.

A matrix $\mathbb{M} := \langle \mathbf{A}, D \rangle$ is *monadic* if there is a unary formula $\mathrm{S}(p) \in L_\Sigma(\{p\})$, sometimes called a *separator*, for each pair of truth values $x, y \in A$, such that $\mathrm{S}_{\mathbf{A}}(x) \in D$ and $\mathrm{S}_{\mathbf{A}}(y) \in A \backslash D$ or vice-versa. A multiple-conclusion Hilbert calculus

R is Θ-*analytic*, for $\Theta(p)$ a set of unary formulas, whenever $\Phi \rhd_{\mathsf{R}} \Psi$ is witnessed by a derivation using only subformulas of $\Phi \cup \Psi$ or formulas in $\bigcup_{\gamma \in \mathsf{subf}(\Phi\cup\Psi)} \Theta(\gamma)$. For example, if $\Theta(p) = \{p, \neg p\}$ and R is Θ-analytic, checking whether $r, q \wedge r \rhd_{\mathsf{R}} p \to r, \neg q$ amounts to looking only for derivations in which formulas in $\{r, q, p, q \wedge r, p \to r, \neg q, \neg r, \neg p, \neg(q \wedge r), \neg(p \to r), \neg\neg q\}$ occur. Because in an OL-matrix p separates $(\mathbf{1}, \mathbf{0})$ and $(\mathbf{1/2}, \mathbf{0})$, while $\neg p$ separates $(\mathbf{1}, \mathbf{1/2})$, the following holds.

Lemma 1. *Any* OL-*matrix over* $\Sigma \supseteq \{\neg\}$ *is monadic, with set of separators* $\{p, \neg p\}$.

Lemma 1 implies that all multiple-conclusion fragments/expansions of sOL containing negation admit $\{p, \neg p\}$-analytic multiple-conclusion Hilbert-style axiomatizations generated from the matrix description in a modular way, as the next theorem states. For a detailed presentation of the calculi generation for three-valued logics and for the associated adequacy proofs, see [9, Sec. 5].

Theorem 1. *Let* Σ *be a signature such that* $\neg \in \Sigma$ *and* $\mathbb{M}$ *be an* OL-*matrix over* Σ. *Then* $\rhd_{\mathbb{M}}$ *is axiomatized by the* $\{p, \neg p\}$-*analytic calculi*

$$\mathsf{R}_\Sigma := \bigcup\nolimits_{© \in \Sigma} \mathsf{R}_{©},$$

where each set $\mathsf{R}_{©}$, $© \in \Sigma$, *is generated by the procedure described in [9]. For the connectives of* OL, *these sets are displayed in Figure 3.*

$\mathsf{R}_\neg$

$$\frac{p}{\neg\neg p}\mathsf{r}_1^{\neg} \qquad \frac{\neg\neg p}{p}\mathsf{r}_2^{\neg} \qquad \frac{}{p, \neg p}\mathsf{r}_3^{\neg}$$

$\mathsf{R}_\to$

$$\frac{p, p \to q}{q}\mathsf{r}_1^{\to} \qquad \frac{q}{p \to q}\mathsf{r}_2^{\to} \qquad \frac{}{p, p \to q}\mathsf{r}_3^{\to}$$

$$\frac{p, \neg(p \to q)}{\neg q}\mathsf{r}_4^{\to} \qquad \frac{\neg q}{\neg(p \to q)}\mathsf{r}_5^{\to} \qquad \frac{}{p, \neg(p \to q)}\mathsf{r}_6^{\to}$$

$\mathsf{R}_\vee$

$$\frac{}{\neg p, p \vee q}\mathsf{r}_1^{\vee} \qquad \frac{}{\neg q, p \vee q}\mathsf{r}_2^{\vee} \qquad \frac{\neg(p \vee q)}{\neg p}\mathsf{r}_3^{\vee} \qquad \frac{\neg(p \vee q)}{\neg q}\mathsf{r}_4^{\vee} \qquad \frac{\neg(p \vee q), p \vee q}{p}\mathsf{r}_5^{\vee}$$

$$\frac{\neg(p \vee q), p \vee q}{q}\mathsf{r}_6^{\vee} \qquad \frac{\neg p, \neg q}{\neg(p \vee q)}\mathsf{r}_7^{\vee} \qquad \frac{\neg p, p \vee q}{q}\mathsf{r}_8^{\vee} \qquad \frac{\neg q, p \vee q}{p}\mathsf{r}_9^{\vee} \qquad \frac{p \vee q}{p, q}\mathsf{r}_{10}^{\vee} \qquad \frac{p, q}{p \vee q}\mathsf{r}_{11}^{\vee}$$

$\mathsf{R}_\wedge$

$$\frac{p \wedge q}{p}\mathsf{r}_1^{\wedge} \qquad \frac{p \wedge q}{q}\mathsf{r}_2^{\wedge} \qquad \frac{p, q}{p \wedge q}\mathsf{r}_3^{\wedge}$$

$$\frac{p \wedge q, \neg(p \wedge q)}{\neg p}\mathsf{r}_4^{\wedge} \qquad \frac{p \wedge q, \neg(p \wedge q)}{\neg q}\mathsf{r}_5^{\wedge} \qquad \frac{\neg p, \neg q}{\neg(p \wedge q)}\mathsf{r}_6^{\wedge} \qquad \frac{p}{p \wedge q, \neg q}\mathsf{r}_7^{\wedge} \qquad \frac{q}{p \wedge q, \neg p}\mathsf{r}_8^{\wedge}$$

Fig. 3. Multiple-conclusion rules for the connectives of sOL.

For examples of derivations in the multiple-conclusion calculus for sOL obtained from the above theorem, see Figure 4. The next theorem presents axiomatizations (recall, not substitution-invariant) for the corresponding multiple-conclusion fragments and expansions of OL.

Theorem 2. *Let* Σ *be a signature such that* $\neg \in \Sigma$ *and* $\mathbb{M}$ *be an* OL-*matrix over* Σ. *Then* $\rhd_{\mathbb{M}}^{\mathsf{biv}}$ *is axiomatized by*

$$\mathsf{R}_\Sigma^{\mathrm{OL}} := \mathsf{R}_\Sigma \cup \left\{ \frac{p, \neg p}{\varnothing}[p] : p \in P \right\}.$$

Proof. The nontrivial inclusion is $\rhd^{\mathsf{biv}}_{\mathbb{M}} \subseteq \rhd_{\mathsf{R}^{\mathrm{OL}}_{\Sigma}}$. Following the standard completeness proofs exemplified in [9, Sec. 5.2], from $\Phi \blacktriangleright_{\mathsf{R}^{\mathrm{OL}}_{\Sigma}} \Psi$ we construct a valuation v witnessing $\Phi \blacktriangleright^{\mathsf{biv}}_{\mathbb{M}} \Psi$. In that construction, the rules $\frac{p,\neg p}{\varnothing}[p]$ force that the valuation only assigns values in $\{\mathbf{0},\mathbf{1}\}$ to propositional variables and the fact that $\mathsf{R}_{\Sigma} \subseteq \mathsf{R}^{\mathrm{OL}}_{\Sigma}$ and R_{Σ} axiomatizes $\rhd_{\mathbb{M}}$ guarantees the homomorphism requirement on v.

The reader may appreciate the importance of the extra rules in the above theorem by proving $p \vee (q \to r) \vdash_{\mathrm{OL}} p \vee r$, even though $p \vee (q \to r) \nvdash_{\mathrm{sOL}} p \vee r$ (easy to see semantically). More examples like this may be found in [5].

3.2 Single-conclusion

When a suitable connective is available, a multiple-conclusion calculus may be effectively translated into a single-conclusion calculus for its single-conclusion companion [13]. We now define what is a suitable connective and the calculi translations.

Definition 1. *Let $\vdash$ be a single-conclusion logic over $L_{\Sigma}(P)$ and © be a derived connective in $L_{\Sigma}(P)$. Then © is*

1. *a* disjunction *in $\vdash$ whenever $\Gamma, \varphi © \psi \vdash \gamma$ iff $\Gamma, \varphi \vdash \gamma$ and $\Gamma, \psi \vdash \gamma$, for all $\Gamma \cup \{\varphi, \psi, \gamma\} \subseteq L_{\Sigma}(P)$.*
2. *an* implication *in $\vdash$ whenever $\Gamma \vdash \varphi © \psi$ iff $\Gamma, \varphi \vdash \psi$, for all $\Gamma \cup \{\varphi, \psi\} \subseteq L_{\Sigma}(P)$.*

Note that, in standard terminology, an implication in $\vdash$ is a binary connective satisfying the Deduction-Detachment Theorem (DDT). In what follows, given a set of formulas Φ and a binary connective $\oplus$, let $\Phi \oplus \psi := \{\varphi \oplus \psi \mid \varphi \in \Phi\}$ and $\bigoplus\{\varphi_1, \ldots, \varphi_m\} := \varphi_1 \oplus (\varphi_2 \oplus \ldots (\ldots \oplus \varphi_n) \ldots)$.

Definition 2. *Let R be a multiple-conclusion calculus and $\oplus$ be a binary connective. We define $\mathsf{R}^{\oplus}$ as the single-conclusion calculus*

$$\left\{\frac{p \oplus p}{p}, \frac{p}{p \oplus q}, \frac{p \oplus q}{q \oplus p}, \frac{p \oplus (q \oplus r)}{(p \oplus q) \oplus r}\right\} \cup \left\{\mathsf{r}^{\oplus} \mid \mathsf{r} \in \mathsf{R}\right\}$$

where $\mathsf{r}^{\oplus}$ is $\frac{\varnothing}{\bigoplus \Psi}[\Pi]$ if $\mathsf{r} = \frac{\varnothing}{\Psi}[\Pi]$, $\frac{\Phi \oplus p_0}{(\bigoplus \Psi) \oplus p_0}[\Pi]$ if $\mathsf{r} = \frac{\Phi}{\Psi}[\Pi]$, and $\frac{\Phi \oplus p_0}{p_0}[\Pi]$ if $\mathsf{r} = \frac{\Phi}{\varnothing}[\Pi]$, for p_0 a propositional variable not occurring in r.

Now we move to translations when an implication is present. Let p_0 be a propositional variable not occurring in $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$ and $\multimap$ be a binary connective. Define $\{\varphi_1, \ldots, \varphi_m\} \multimap \{\psi_1, \ldots, \psi_n\} := \varphi_1 \multimap (\{\varphi_2, \ldots, \varphi_m\} \multimap \{\psi_1, \ldots, \psi_n\})$, $\varnothing \multimap \{\psi_1, \ldots, \psi_n\} := (\psi_1 \multimap p_0) \multimap (\varnothing \multimap \{\psi_2, \ldots, \psi_n\})$ and $\varnothing \multimap \varnothing := p_0$. For example, $\{p, q\} \multimap \{p \wedge q\} = p \multimap (q \multimap (((p \wedge q) \multimap p_0) \multimap p_0))$.

Definition 3. *Let R be a multiple-conclusion calculus and $\multimap$ be a binary connective. We define $\mathsf{R}^{\multimap}$ as the single-conclusion calculus containing all rules and axioms of intuitionistic implication (where $\multimap$ is taken as this implication) and axioms of the form*

$$\frac{}{\{\varphi_1, \ldots, \varphi_m\} \multimap \{\psi_1, \ldots, \psi_n\}}[\Pi]$$

for each rule $\frac{\varphi_1, \ldots, \varphi_m}{\psi_1, \ldots, \psi_n}[\Pi]$ belonging to R.

The following theorem establishes that, when $\oplus$ and $\multimap$ are respectively a disjunction and an implication as previously defined, the above translations produce (finite) single-conclusion axiomatizations from (finite) multiple-conclusion axiomatizations. In its original formulation, rules of the general form $\frac{\Phi}{\Psi}[\Pi]$ were not considered, but their addition does not invalidate the result.

Theorem 3 ([13, Thm. 5.37, Lem. 19.20]). *Let* R *be a multiple-conclusion calculus. (1) If* $\oplus$ *is a disjunction in* $\vdash_{\mathsf{R}}$*, then* $\vdash_{\mathsf{R}} = \vdash_{\mathsf{R}^{\oplus}}$*. (2) If* $\rightarrowtriangle$ *is an implication in* $\vdash_{\mathsf{R}}$*, then* $\vdash_{\mathsf{R}} = \vdash_{\mathsf{R}^{\rightarrowtriangle}}$*.*

It is not hard to check that $\to$ is an implication in sOL (see also Section 4); this gives us the following single-conclusion axiomatizations.

Theorem 4. *Let* Σ *be a signature such that* $\{\neg, \to\} \subseteq \Sigma$*, and let* $\mathbb{M}$ *be an* OL*-matrix over* Σ*. Then* $\vdash_{\mathbb{M}}$ *is axiomatized by* $\mathsf{R}_{\Sigma}^{\to}$*, where* R_{Σ} *is given as in Theorem 1.*

More interestingly, we may replace $\to$ by either $\land$ or $\lor$, providing axiomatizations for more fragments of sOL. Neither $\land$ nor $\lor$ is a disjunction or an implication in the above sense, but each of them allows us to define a connective that is a disjunction and thus suitable for the multiple-conclusion to single-conclusion translation.

Theorem 5. *Let* Σ *be a signature such that either* $\{\neg, \land\} \subseteq \Sigma$ *or* $\{\neg, \lor\} \subseteq \Sigma$ *and let* $\mathbb{M}$ *be an* OL*-matrix over* Σ*. Then* $\vdash_{\mathbb{M}}$ *is axiomatized by* $\mathsf{R}_{\Sigma}^{\curlyvee}$*, where* R_{Σ} *is given as in Theorem 1, and* $p \curlyvee q := (p \Rightarrow q) \Rightarrow ((q \Rightarrow p) \Rightarrow p)$*.*

Proof. Let $p \curlyvee q := (p \Rightarrow q) \Rightarrow ((q \Rightarrow p) \Rightarrow p)$ and recall that $\Rightarrow$ was defined as $\neg p \lor q$ (thus using only $\neg$ and $\lor$). Note that it could also have been defined using $\neg$ and $\land$, since $\lor$ is definable from $\neg$ and $\land$. From the truth table of $\curlyvee$ displayed in Figure 1, it is easy to see that $\curlyvee$ is a disjunction in the sense of Definition 1.

We display in Figure 5 the axiomatization of sOL following the previous theorem.

Remark 1. We could have used the connective $\sqcup$ defined before instead of $\curlyvee$.

Remark 2. Since $\curlyvee$ was defined using only $\Rightarrow$, this result (and the next) also applies to the term-definable single-conclusion fragments $\{\neg, \Rightarrow\}$ of OL and sOL.

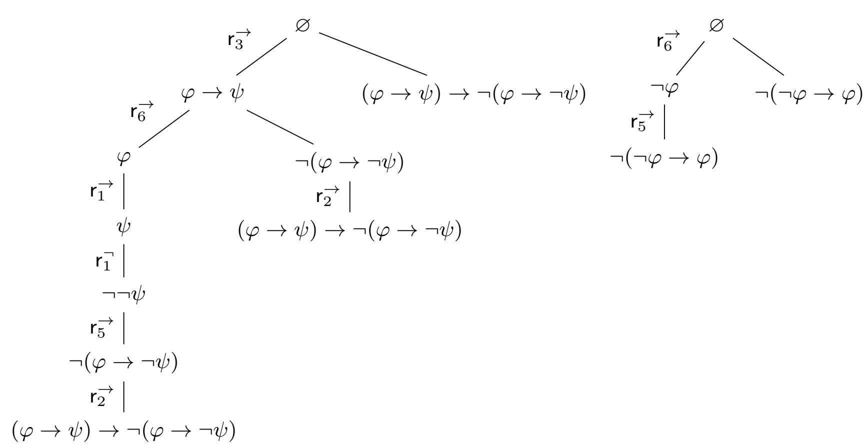

Fig. 4. Derivations of Boethius' thesis and Aristotle's thesis in sOL (see Figure 3).

With the above results, we axiomatized via single-conclusion calculi all single-conclusion fragments/expansions of sOL containing $\neg$. For some of them, and for sOL itself, two calculi were presented. The ones obtained from Theorem 4 have one rule and many axioms, while the ones from Theorem 5 tend to be rich in rules and have only a few axioms. We now proceed to extend these calculi to axiomatize the corresponding fragments and expansions of OL.

Theorem 6. *Let* Σ *be a signature either with* $\{\neg, \to\} \subseteq \Sigma$*,* $\{\neg, \lor\} \subseteq \Sigma$ *or* $\{\neg, \land\} \subseteq \Sigma$*, and let* $\mathbb{M}$ *be an* OL*-matrix over* Σ*. Then* $\vdash_{\mathbb{M}}^{\mathsf{biv}} = \vdash_{(\mathsf{R}_{\Sigma}^{\mathrm{OL}})^{\copyright}}$*, where* $\copyright = \to$ *if* $\to \in \Sigma$ *and* $\copyright = \curlyvee$ *otherwise.*

$\mathsf{R}_{\curlyvee}$

$$\frac{p \curlyvee p}{p} \qquad \frac{p}{p \curlyvee q} \qquad \frac{p \curlyvee q}{q \curlyvee p} \qquad \frac{p \curlyvee (q \curlyvee r)}{(p \curlyvee q) \curlyvee r}$$

$\mathsf{R}_{\neg}$

$$\frac{p \curlyvee r}{\neg\neg p \curlyvee r} \qquad \frac{\neg\neg p \curlyvee r}{p \curlyvee r} \qquad \frac{}{p \curlyvee \neg p}$$

$\mathsf{R}_{\to}$

$$\frac{p \curlyvee r, (p \to q) \curlyvee r}{q \curlyvee r} \qquad \frac{q \curlyvee r}{(p \to q) \curlyvee r} \qquad \frac{}{p \curlyvee (p \to q)}$$

$$\frac{p \curlyvee r, \neg(p \to q) \curlyvee r}{\neg q \curlyvee r} \qquad \frac{\neg q \curlyvee r}{\neg(p \to q) \curlyvee r} \qquad \frac{}{p \curlyvee \neg(p \to q)}$$

$\mathsf{R}_{\vee}$

$$\frac{}{\neg p \curlyvee (p \vee q)} \qquad \frac{}{\neg q \curlyvee (p \vee q)} \qquad \frac{\neg(p \vee q) \curlyvee r}{\neg p \curlyvee r} \qquad \frac{\neg(p \vee q) \curlyvee r}{\neg q \curlyvee r}$$

$$\frac{\neg(p \vee q) \curlyvee r, (p \vee q) \curlyvee r}{p \curlyvee r} \qquad \frac{\neg(p \vee q) \curlyvee r, (p \vee q) \curlyvee r}{q \curlyvee r} \qquad \frac{\neg p \curlyvee r, \neg q \curlyvee r}{\neg(p \vee q) \curlyvee r}$$

$$\frac{\neg p \curlyvee r, (p \vee q) \curlyvee r}{q \curlyvee r} \qquad \frac{\neg q \curlyvee r, (p \vee q) \curlyvee r}{p \curlyvee r} \qquad \frac{(p \vee q) \curlyvee r}{(p \curlyvee q) \curlyvee r} \qquad \frac{p \curlyvee r, q \curlyvee r}{(p \vee q) \curlyvee r}$$

$\mathsf{R}_{\wedge}$

$$\frac{(p \wedge q) \curlyvee r}{p \curlyvee r} \qquad \frac{(p \wedge q) \curlyvee r}{q \curlyvee r} \qquad \frac{p \curlyvee r, q \curlyvee r}{(p \wedge q) \curlyvee r}$$

$$\frac{(p \wedge q) \curlyvee r, \neg(p \wedge q) \curlyvee r}{\neg p \curlyvee r} \qquad \frac{(p \wedge q) \curlyvee r, \neg(p \wedge q) \curlyvee r}{\neg q \curlyvee r}$$

$$\frac{\neg p \curlyvee r, \neg q \curlyvee r}{\neg(p \wedge q) \curlyvee r} \qquad \frac{p \curlyvee r}{((p \wedge q) \curlyvee \neg q) \curlyvee r} \qquad \frac{q \curlyvee r}{((p \wedge q) \curlyvee \neg p) \curlyvee r}$$

Fig. 5. Single-conclusion calculus for single-conclusion sOL produced via Definition 2. By Theorem 5, one can modularly add suitable rules to $\mathsf{R}_{\curlyvee} \cup \mathsf{R}_{\neg} \cup \mathsf{R}_{\vee}$ to axiomatize fragments/expansions of sOL over signatures $\Sigma \supseteq \{\neg, \vee\}$ ($\vee$ may be replaced by $\wedge$).

4 Algebraic semantics

We denote by $\mathbf{O}_3$ the three-element algebra whose operation tables are given in Figure 1, viewed as an algebra in the language $\{\wedge, \vee, \to, \neg\}$. Denote by $\mathbb{OL}$ the quasi-variety generated by $\mathbf{O}_3$ (as we shall prove, $\mathbb{OL}$ is in fact a variety), and let $\vDash_{\mathbb{OL}}$ denote the corresponding relative equational consequence relation.

Algebraizability of sOL. The matrix semantics of sOL makes it easy to check that sOL is algebraizable in the sense of Blok and Pigozzi [1]. In what follows, abbreviate $|\alpha| := \alpha \Rightarrow \alpha$.

Theorem 7. sOL *is algebraizable with translations* $\tau \colon x \mapsto x \approx |x|$ *and* $\rho \colon \varphi \approx \psi \mapsto \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$. *(Alternatively, one may take* $\tau \colon x \mapsto x \approx x \to x$ *and* $\rho \colon \varphi \approx \psi \mapsto \{\varphi \to \psi, \psi \to \varphi, \neg\varphi \to \neg\psi, \neg\psi \to \neg\varphi\}$*.)*

Proof. Observe (using Figure 2) that a valuation v over the matrix of sOL satisfies an equation $\varphi \approx \varphi \Rightarrow \varphi$ iff $v(\varphi) \in \{\mathbf{1/2}, \mathbf{1}\}$. Thus, for all formulas Γ, φ, we have $\Gamma \vdash_{\mathrm{sOL}} \varphi$ iff $\tau(\Gamma) \vDash_{\mathbb{OL}} \tau(\varphi)$. This is condition (ALG1) of algebraizability [8, Def. 3.11]. To establish algebraizability, it remains to prove (ALG4), i.e., that every equation $\varphi \approx \psi$ is inter-derivable in $\vDash_{\mathbb{OL}}$ with $\tau(\rho(\varphi \approx \psi))$, which is the set $\{\varphi \Rightarrow \psi \approx |\varphi \Rightarrow \psi|, \psi \Rightarrow \varphi \approx |\psi \Rightarrow \varphi|\}$. This is easily verified in $\mathbf{O}_3$.

Algebraic counterpart of sOL. By inspection, $\mathbf{O}_3$ (Figure 1) suggests that:

1. The tables of $\sqcap, \sqcup$ are precisely those of the conjunction and disjunction of strong Kleene logic and of G. Priest's Logic of Paradox, both defined over the language $\{\sqcap, \sqcup, \neg\}$ (see [9] for further background on these logics).
2. Thus, the Logic of Paradox (which also has $\{\mathbf{1/2}, \mathbf{1}\}$ as designated set, whereas strong Kleene has $\{\mathbf{1}\}$ alone) may be viewed as a definable subsystem of sOL.
3. If we consider the language $\{\sqcap, \sqcup, \supset, \neg\}$ then we have the connectives of Da Costa and D'Ottaviano's three-valued logic $\mathcal{J}3$ (whose designated set is also $\{\mathbf{1/2}, \mathbf{1}\}$) minus the truth constants. These, which are otherwise not definable in sOL, need to be explicitly included in the language; we may then define the modal operator of $\Diamond$ of $\mathcal{J}3$ by $\Diamond x := x \wedge \mathbf{1}$.
4. We note that, if we add either $\mathbf{1}$ or $\mathbf{0}$ to the language of sOL, then every possible three-valued connective becomes definable. This is a consequence of the observation that $\mathbf{O}_3$ then becomes a primal algebra (Theorem 8).

The following proposition is also a matter of straightforward computations.

Proposition 1. *Let $\mathbf{O}_3 = \langle O_3; \wedge, \vee, \rightarrow, \neg \rangle$ be endowed with the above-defined operations (Figure 2).*

1. *$\langle O_3; \wedge, \mathbf{1/2}, \mathbf{0} \rangle$ is a meet semilattice with $\mathbf{1/2}$ as maximum and $\mathbf{0}$ as minimum.*
2. *$\langle O_3; \vee, \mathbf{1}, \mathbf{1/2} \rangle$ is a join semilattice with $\mathbf{1}$ as maximum and $\mathbf{1/2}$ as minimum.*
3. *$\langle O_3; \sqcap, \sqcup, \mathbf{1}, \mathbf{0} \rangle$ is a lattice with $\mathbf{1}$ as maximum and $\mathbf{0}$ as minimum.*

For all unexplained universal algebraic terminology, we refer the reader to [2].

Theorem 8. *Denote by $V(\mathbf{O}_3)$ the variety generated by $\mathbf{O}_3$.*

1. *$V(\mathbf{O}_3)$ is both congruence-distributive and congruence-permutable (i.e., it is an arithmetical variety).*
2. *$V(\mathbf{O}_3) = \mathbb{OL}$.*
3. *$\mathbf{O}_3$ is quasi-primal, hence $\mathbb{OL}$ is a discriminator variety.*
4. *If we add either $\mathbf{0}$ or $\mathbf{1}$ as a constant to $\mathbf{O}_3$, then the latter becomes a primal algebra (where every n-ary function for $n \geq 1$ is representable by a term).*

Proof. 1. Congruence-distributivity follows from the observation that $\mathbf{O}_3$ has a term-definable lattice structure (item (iii) of Proposition 1). Congruence-permutability is witnessed by the Maltsev term $p(x, y, z)$ defined as follows (cf. [11, Thm. 4.10]): $p(x, y, z) := (((x \Rightarrow y) \sqcap (z \Rightarrow z)) \Rightarrow z) \sqcap (((z \Rightarrow y) \sqcap (x \Rightarrow x)) \Rightarrow x)$.

2. Since $V(\mathbf{O}_3)$ is congruence-distributive, by item (iii) of [4, Thm. 3.6]) it suffices to verify that $HS(\mathbf{O}_3) \subseteq IS(\mathbf{O}_3)$, which is very easy ($\mathbf{O}_3$ has only one proper subalgebra with $\{\mathbf{1/2}\}$ as universe, and no non-trivial homomorphic images).

3. Taking into account item (i) and the fact that $\mathbf{O}_3$ is hereditarily simple, apply Pixley's characterization [2, Thm. IV.10.7] to conclude that $\mathbf{O}_3$ is quasiprimal.

4. If we further add one of the non-definable constants ($\mathbf{0}$ or $\mathbf{1}$) to $\mathbf{O}_3$, then by [2, Cor. 10.8] we obtain a primal algebra.

Axiomatizing $\mathbb{OL}$. The algebraizability result (Theorem 7) can be used to obtain a presentation of the quasi-variety $\mathbb{OL}$ in the standard way (see [8, Prop. 3.44]), as well as of the subreducts of $\mathbb{OL}$ corresponding to algebraizable fragments of sOL (e.g. those capable of expressing either $\Rightarrow$ or $\rightarrow$ and $\neg$, relying on the axiomatizations obtained in Theorems 4–6). Moreover, one may obtain a more standard Hilbert presentation for sOL by directly proving that the logic determined by the following calculus is algebraizable (with the same translations τ, ρ of Theorem 7), and that its equivalent semantics is $\mathbb{OL}$. This is straightforward, but requires a number of derivations that we omit due to space limitations. The calculus is given by the following axiom schemata, with *modus ponens* (from φ and $\varphi \rightarrow \psi$ infer ψ) as the only inference rule (we abbreviate $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$):

(HOL1)	$\varphi \to (\psi \to \varphi)$
(HOL2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$
(HOL3)	$((\varphi \to \psi) \to \varphi) \to \varphi$
(HOL4)	$(\varphi \wedge \psi) \to \varphi$
(HOL5)	$(\varphi \wedge \psi) \to \psi$
(HOL6)	$(\varphi \to \psi) \to ((\varphi \to \gamma) \to (\varphi \to (\psi \wedge \gamma)))$
(HOL7)	$\neg\neg\varphi \leftrightarrow \varphi$
(HOL8)	$(\varphi \to \neg\varphi) \to \neg\varphi$
(HOL9)	$((\varphi \to \neg\psi) \wedge (\psi \to \neg\varphi)) \leftrightarrow \neg(\varphi \wedge \psi)$.
(HOL10)	$\neg(\varphi \to \psi) \leftrightarrow (\varphi \to \neg\psi)$

(The 'H' in (HOL1) etc. refers to 'Hilbert'.) We note that the only non-classically valid scheme is (HOL10), while (HOL1)–(HOL6) constitute, with *modus ponens*, an axiomatization of the conjunction-implication fragment of classical logic. This entails that every classical tautology in this language is derivable in HOL. Also observe that, since *modus ponens* is the only rule of inference, axioms (HOL1) and (HOL2) give us that $\to$ satisfies the DDT (see Definition 1 (2)).

For a quasi-equational presentation of $\mathbb{OL}$, thus, we may employ the following quasi-equations (cf. [8, Prop. 3.44]):

1. $\alpha \approx |\alpha|$ for each axiom α in (HOL 1)–(HOL 9),
2. if $\alpha \approx |\alpha|$ and $\alpha \to \beta \approx |\alpha \to \beta|$, then $\beta \approx |\beta|$,
3. if $\alpha \Rightarrow \beta \approx |\alpha \Rightarrow \beta|$ and $\beta \Rightarrow \alpha \approx |\beta \Rightarrow \alpha|$, then $\alpha \approx \beta$.

5 Future work

Having axiomatized the logic of ordinary discourse OL and investigated the logico-algebraic features of its structural counterpart (sOL), we believe to have contributed to the advancement of the study of connexive (multiple- and single-conclusion) logics. We view the present study as yet another vindication of the usefulness of multiple-conclusion calculi in the study of finite-valued logics. Beyond the results presented here, we speculate that the following directions may prove fruitful in future research.

1. Due to space limitations, the algebraic aspects of sOL have been touched only sketchily in the present paper. We reserve a more comprehensive study – including a proof of algebraizability of the Hilbert calculus introduced in Section 4, a more perspicuous presentation of the variety $\mathbb{OL}$, etc. – to a future publication [12].

2. The papers [6,7] by P. Egré *et al.* contain an extensive discussion of three-valued logics that model conditionals in natural language. Among other systems, the authors consider a variant of sOL (denoted CC/TT) that employs the implication $\to$ together with the connectives $\sqcap, \sqcup$ instead of the primitive conjunction and disjunction of sOL. Present space limitations do not allow us to compare in detail our approach with that of Egré *et al.*, so this issue too will have to be left for future research. However, we may anticipate that our algebraic analysis of sOL throws some light on the observations of [7, Sec. 4], in particular the fact that a translation $\rho\colon \varphi \approx \psi \mapsto \{\varphi \to \psi, \psi \to \varphi\}$ does not guarantee algebraizability, either for sOL or for CC/TT (cf. [7, Lemma 4.18]). On the other hand, it is easy to see that the translation considered in Section 4, namely $\rho\colon \varphi \approx \psi \mapsto \{\varphi \to \psi, \psi \to \varphi, \neg\varphi \to \neg\psi, \neg\psi \to \neg\varphi\}$, guarantees algebraization for both logics, thereby settling the problem left open in [7].

3. As mentioned earlier, sOL is definitionally equivalent to an expansion of Da Costa and D'Ottaviano's logic $\mathcal{J}3$, which is in turn an axiomatic extension of paraconsistent Nelson logic. This suggests that $\mathbb{OL}$ may be viewed as a subvariety of

N4-lattices and, as such, its members may be given a twist-structure representation. Developing such a study may not only provide further insight into $\mathbb{OL}$, but also clarify the relationship between sOL and other related non-classical systems, such as the connexive logic C [14].

4. It might be interesting to develop a study similar to the present one for other logics defined from the algebra $\mathbf{O}_3$ with different sets of designated values, e.g. $\{\mathbf{1}\}$ or $\{\mathbf{1/2}\}$, or the order-preserving logics associated to the orderings naturally arising on $\mathbf{O}_3$ (cf. Proposition 1). An obviously different but related question is whether any of these systems admit an interpretation in line with Cooper's original proposal of formalizing reasoning in ordinary discourse.

5. An algebraic study of the (term-definable) fragments of sOL axiomatized in Section 2 also appears to be promising. Some of these – such as the $\{\to, \neg\}$-fragment and the $\{\wedge, \neg\}$-fragment – are algebraizable, suggesting that they may be easily treatable with algebraic methods. The $\{\Rightarrow\}$-fragment, not considered here, is also easily seen to be algebraizable, and may be axiomatized by the methods employed in the present study. By [13], every multiple-conclusion finite-valued logic is axiomatized by a finite multiple-conclusion calculus; given that $\curlyvee$ was defined using only $\Rightarrow$, a single-conclusion calculus for this fragment exists by Theorem 3.

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