The Suitability of Upper Boundary Algebra for Solving Partial Fuzzy Relational Equations*

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Abstract. Three-valued logics became a classical topic for logicians and not surprisingly, they were extended to partial fuzzy logics that allow modeling distinct types of undefined truth-values, i.e., values, that are neither true nor false, even in the graded sense. Such logics and related algebras may model reasoning with non-denoting terms, missing or unknown values, and other interesting cases. However, in order to be able to model real cases, the algebraic models need to be mirrored in applied tools such as inference systems. Therefore, the investigation of partial fuzzy relational equations that question the most natural property of such systems is a straightforward step. This step has been already made, however, satisfactory results were obtained only for the direct product inference (compositional rule of inference). Intuitively, this was due to the application of partial algebras that employ the so-called lower boundary strategy. This article introduces upper boundary algebraic strategy and shows, that equally satisfactory results may be obtained also for the Bandler-Kohout subproduct.

Keywords: Upper boundary algebra · Undefined values · Partial algebra · Partial fuzzy set theory · Inference systems · Systems of fuzzy relational equations · Bandler-Kohout subproduct.

1 Introduction and motivation

Three-valued logic, pioneered in the 1920s by Łukasiewicz [23], appeared as a conceptual framework for handling statements that disobey the binary categorization of truth and false. Such statements may appear to be "inconsistent", "irrelevant", "meaningless," or "missing". The truth values associated with these statements are *undefined* and mathematically, represented by a dummy value conventionally symbolized as $\star$. This $\star$ is the third value extending the traditional true (1) and false (0) which gives the name to the three-values logics.

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Over the decades, numerous logicians have delved into this subject, producing a range of valuable results, including theoretical advancements [1, 4, 9, 18, 21, 30] and practical applications [20, 31]. Several partial algebras have been created in three-valued logics, each designed for specific purposes and perspectives, aligning with applications or particular interpretations of $\star$. Prominent examples include Bochvar, Sobociński, Kleene, Nelson, and Łukasiewicz algebras.

The generalization of three-valued logics to partial fuzzy logics and related algebraic structures in partial fuzzy set theory has received substantial attention [5, 7, 11, 12, 26]. Numerous partial algebras, including well-known ones, have been generalized within this framework. The introduction of the *Lower estimation algebra* [13] and *Dragonfly algebra* [32], designed with a lower boundary strategy, benefit in practical classification problems such as Dragonfly and Amphibian classification [32]. These algebras are designed to address the missing values, guided by the lower boundary strategy. It means that the algebraic operations attempt to yield the result (truth-value) that we may guarantee from below no matter what would be the real truth value that would replace the missing value represented by $\star$.

We only briefly recall that the development of partial fuzzy algebras enabled the exploration of topics such as compositions of partial fuzzy relations, preservation of residuated lattice properties, and the study of solvability in partial fuzzy relational equations [16, 17] or contributed to the qualitative integrals [19] and free quantification in four-valued logics [6].

From the above-mentioned applications and directions of further development, the primary motivation for this contribution relates to the solvability of partial fuzzy relational equations. Recent research [16] has yielded promising results. It focused on the solvability conditions of two systems, the one with a direct product, $A_i \circ_\theta R = B_i$, and the one with a subdirect product, $A_i \lhd_\theta R = B_i$, where, θ denotes a particular partial algebra, and antecedents A_i as well as consequents B_i are partial fuzzy sets. The investigation revealed that the system with the direct product $\circ_\theta$ provided desirable results especially when the Lower estimation and Dragonfly algebras were considered. This prompts consideration of the compatibility of these two algebras of lower boundary operations with such a system, see also [15].

Motivated by the above-mentioned compatibility in the case of the direct product, we develop a new set of operations to produce similarly positive results for the system with the subdirect product $\lhd$ (also Bandler-Kohout subproduct). Given the duality of both products [8], and consequently, of both systems, our approach is based on constructing an algebra that dually employs a sort of upper boundary strategy.

Let us note that although the whole motivation may seem to be purely mathematical, abstract, and theoretical, its practical impact may be huge. Note that the fuzzy relational compositions serve as a basis for distinct applications, e.g., classification based on expert knowledge encoded in a fuzzy relation. This was the very first case of the use of fuzzy relational compositions, particularly, in medical diagnosis [2, 3] as well as more recent cases of biological species classifications

[14, 32]. Even more often, the applications of fuzzy relational compositions appear in fuzzy inference systems, i.e., in approximate reasoning and automatic deduction, generally speaking. In all such applications, it is often the case that some information is missing. One can name, e.g., the case of questionnaires with non-relevant questions for a certain subset of probands and subsequent use of distinct "N/A" symbols.

Before approaching particular applications and data sets, firm formal grounds have to be determined, theoretical questions have to be investigated, and crucial properties need to be preserved. And this is also the goal of this article that connects fuzzy relational systems and partial fuzzy set theory which is a purely theoretical step, however, with a potentially strong practical impact.

Let us close the Introduction by providing the readers with a brief overview of the structure of the paper. Secion 2 recalls fundamentals of the two lower boundary strategy algebras; Section 3 introduces the dual upper boundary algebraic approach; Section 4 provides the core investigation on the solvability of partial fuzzy relational equations and finally, Section 5 closes the paper with the discussion.

2 Lower estimation algebra and Dragonfly algebra

This section briefly recalls the Lower estimation (Le) algebra [13] and the Dragonfly (D) algebra [32]. We fix the underlying algebra as a complete residuated lattice $\langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$. Furthermore, let $[0,1]^\star = [0,1] \cup \{\star\}$ be the set of the truth-values extended by dummy value $\star$. All the binary operations $\otimes_\theta, \wedge_\theta, \vee_\theta, \rightarrow_\theta$ where $\theta \in \{\mathrm{Le}, \mathrm{D}\}$ operate on $[0,1]^\star$ and their definition is provided in Table 1.

Table 1. Truth table of Lower estimation and Dragonfly operations, $\theta \in \{\mathrm{Le}, \mathrm{D}\}$.

		$(\otimes_\theta, \wedge_\theta)$	$\vee_\theta$	$\rightarrow_{\mathrm{D}}$	$\rightarrow_{\mathrm{Le}}$
$a \in [0,1]$	$b \in [0,1]$	$(\otimes, \wedge)$	$\vee$	$\rightarrow$	$\rightarrow$
$a \in (0,1)$	$\star$	$\star$	a	$\star$	$\star$
$\star$	$b \in (0,1)$	$\star$	b	b	b
$\star$	$\star$	$\star$	$\star$	1	$\star$
$\star$	1	$\star$	1	1	1
1	$\star$	$\star$	1	$\star$	$\star$
$\star$	0	0	$\star$	$\star$	0
0	$\star$	0	$\star$	1	1

The Lower estimation and the Dragonfly algebras were designed in accordance with the lower boundary approach. This can be seen in the combinations of a value $a \in (0,1)$ and the missing value $\star$ as follows. We observe that the disjunction of a and $\star$ always results in a value higher than or equal to a, regardless of the truth value replaced for $\star$ in the unit interval $[0,1]$. Therefore, the result of $a \vee_\theta \star$, for $\theta \in \{\mathrm{Le}, \mathrm{D}\}$, is estimated from below by a. Conversely, the conjunctive operations $a \otimes_\theta b$ and $a \wedge_\theta b$ result in $\star$. This is due to the unknown lower bound of the conjunction of these values, which can be as low as 0 when $\star$ is lowered down to 0.

The implication operations $\rightarrow_{\mathrm{Le}}, \rightarrow_{\mathrm{D}}$ coincide in most positions. For example, $\star \rightarrow_\theta a = a$, for $a \in (0,1)$ and $\theta \in \{\mathrm{Le}, \mathrm{D}\}$. This result arises from the residuated lattice property $a \rightarrow b \geq b$ and aligns with the lower boundary approach. Indeed, we have to assume that the unknown (unobserved) $\star$ may be replaced by 1 (later on after being observed) and then, we would get $1 \rightarrow a = a$ and so, a is the lowest value we can guarantee independently on the choice of a value that would replace $\star$. On the other hand, we have $a \rightarrow_\theta \star = \star$ for $\theta \in \{\mathrm{Le}, \mathrm{D}\}$ as the guaranteed value is *unknown* which is represented by $\star$. Note, that the implications $\rightarrow_{\mathrm{Le}}$ and $\rightarrow_{\mathrm{D}}$ differ at the positions where $\star$ implies $\star$ and $\star$ implies 0. Specifically, $\star \rightarrow_{\mathrm{Le}} \star = \star$ mimics the behavior of the operations from the Kleene algebra, while $\star \rightarrow_{\mathrm{D}} \star = 1$ to maintain the well-known residuated lattice property that $a \rightarrow b = 1$ if and only if $a \leq b$. For the other position, it is sufficient to note that $\star \rightarrow_{\mathrm{Le}} 0 = 0$ and $\star \rightarrow_{\mathrm{D}} 0 = \star$. Although they differ, both approaches are reasonable in design and effective in practical applications.

We recall two orderings used in the Lower estimation and Dragonfly algebras. The first one is identical to the Kleene ordering $\leq$, i.e., it positions $\star$ between 0 and 1 ($0 \leq \star \leq 1$) however, $\star$ is incomparable to any other truth value $a \in (0,1)$. The second one is the lattice-like ordering $\leq_\ell$ derived from the "meet" and "join" operations $(\wedge_\theta, \vee_\theta)$. In particular, $a \leq_\ell b$ if and only if $a \wedge_\theta b = a$ (and also $a \vee_\theta b = b$). According to $\leq_\ell$, $\star$ is comparable to any value from the unit interval $[0,1]$, in particular, we get the chain ordering $0 \leq_\ell \star \leq_\ell a$ for any $a > 0$.

3 Upper boundary algebra

3.1 Formation

The lower boundary strategies turned out to be useful in the investigations of the solvability of partial fuzzy relational equations with the direct product. However, in combination with the Bandler-Kohout subproduct, this approach did not perform any advantage compared to other partial algebras. This is not surprising if we take into account the duality of the direct product and the Bandler-Kohout subproduct. This duality foreshadows the promising approach that should stem from the formation of another partial algebra that represents the dual strategy to the lower boundary one. This algebra will be called *Upper boundary algebra* and denoted by "Ub".

Definition 1 *Let* $\langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ *be a residuated lattice. The structure* $\langle [0,1]^\star, \wedge_{\mathrm{Ub}}, \vee_{\mathrm{Ub}}, \otimes_{\mathrm{Ub}}, \rightarrow_{\mathrm{Ub}}, 0, 1 \rangle$ *is called* Upper boundary algebra *if the operations* $\otimes_{\mathrm{Ub}}$*,* $\wedge_{\mathrm{Ub}}$*,* $\vee_{\mathrm{Ub}}$*,* $\rightarrow_{\mathrm{Ub}}$ $: [0,1]^\star \times [0,1]^\star \longrightarrow [0,1]^\star$ *are given by Table 2:*

Analogously to the case of the Lower estimation and Dragonfly algebras, the structure of the Upper boundary algebra leads to two orderings. The first one

Table 2. Upper boundary algebra operations.

		$(\otimes_{\mathrm{Ub}}, \wedge_{\mathrm{Ub}})$	$\vee_{\mathrm{Ub}}$	$\to_{\mathrm{Ub}}$
$a \in [0,1]$	$b \in [0,1]$	$(\otimes, \wedge)$	$\vee$	$\to$
$a \in (0,1)$	$\star$	a	$\star$	1
$\star$	$b \in (0,1)$	b	$\star$	b
$\star$	$\star$	$\star$	$\star$	1
$\star$	1	$\star$	1	1
1	$\star$	$\star$	1	$\star$
$\star$	0	0	$\star$	0
0	$\star$	0	$\star$	1

denoted by $\leq$ is inherited from the Kleene algebra and it reflects the position $0 \leq \star \leq 1$ however, $\star$ is incomparable to any $a \in (0,1)$. The second ordering is the lattice-like ordering $\leq_\ell$ that is equally constructed as in both lower boundary algebras and thus, it is reasonable to use the same denotation. For the sake of completeness, let us recall the formal definition. For any $a, b \in [0,1]^\star$, we write that

$$a \leq_\ell b \quad \text{if} \quad a \wedge_{\mathrm{Ub}} b = a \quad \text{and} \quad a \vee_{\mathrm{Ub}} b = b\ . \tag{1}$$

It can be easily verified that $\leq_\ell$ is a partial order relation. Moreover, let us mention that the equality $a = b$ for $a, b \in [0,1]^\star$ can be deduced either through $a \leq b$ and $b \leq a$ or through $a \leq_\ell b$ and $b \leq_\ell a$. Additionally, $\leq_\ell$ organizes the elements in $[0,1]^\star$ such that $a \leq_\ell \star \leq_\ell 1$ for any $a < 1$.

3.2 Properties

This section provides readers with several fundamental yet useful properties that will be used in the latter. They are well-known in residuated lattices however, their preservation for the Upper boundary algebra needs to be confirmed.

Proposition 1 *For any $a, b, c \in [0,1]^\star$:*

$$a \wedge_{\mathrm{Ub}} b \leq_\ell a \tag{2}$$
$$a \leq_\ell a \vee_{\mathrm{Ub}} b \tag{3}$$
$$b \leq_\ell a \to_{\mathrm{Ub}} (a \otimes_{\mathrm{Ub}} b) \tag{4}$$
$$a \leq_\ell b,\ a \leq_\ell c \ \Rightarrow\ a \leq_\ell b \wedge_{\mathrm{Ub}} c. \tag{5}$$

Proof: Consider (2) and let $a = \star$. If $b = 1$, $a \wedge_{\mathrm{Ub}} b = \star = a$. If $b \neq 1$, $a \wedge_{\mathrm{Ub}} b = b \leq_\ell \star$. Let $b = \star$ and $a \neq \star$. Then, it holds for $a = 1$ that $a \wedge_{\mathrm{Ub}} b = \star \leq_\ell a$. For $a \neq 1$, we get $a \wedge_{\mathrm{Ub}} b = a$.

Consider (3) and let $a = \star$. If $b = 1$ then $a \vee_{\mathrm{Ub}} b = 1$ and the inequality $\star \leq_\ell 1$ holds. If $b \neq 1$, $a \vee_{\mathrm{Ub}} b = \star = a$ and the inequality is preserved too. Now, consider $b = \star$. If $a = 1$ then $a \vee_{\mathrm{Ub}} b = 1 = a$. If $a \neq 1$ then $a \vee_{\mathrm{Ub}} b = \star$ and thus, $a \leq_\ell \star$.

Consider (4) and let $a = \star$. If $b \in \{\star, 1\}$, the right-hand side of the property equals $\star \rightarrow_{\mathrm{Ub}} (\star \otimes_{\mathrm{Ub}} b) = \star \rightarrow_{\mathrm{Ub}} \star = 1$. If $b \notin \{\star, 1\}$, the right-hand side of the property is $\star \rightarrow_{\mathrm{Ub}} (\star \otimes_{\mathrm{Ub}} b) = \star \rightarrow_{\mathrm{Ub}} b = b$. Now, let $b = \star$. If $a = 1$, the right-hand side of the property is $1 \rightarrow_{\mathrm{Ub}} (1 \otimes_{\mathrm{Ub}} \star) = 1 \rightarrow_{\mathrm{Ub}} \star = \star$. If $a \in [0, 1)$ then we obtain $a \rightarrow_{\mathrm{Ub}} (a \otimes_{\mathrm{Ub}} \star) = a \rightarrow_{\mathrm{Ub}} a = 1$ and thus, $b \leq_\ell 1$ is trivially preserved.

Consider (5) and let $a = \star$. Then $b, c \in \{\star, 1\}$ and thus, $b \wedge_{\mathrm{Ub}} c \in \{\star, 1\}$. Hence, $\star \leq_\ell b \wedge_{\mathrm{Ub}} c$ holds. The case $a \in \{0, 1\}$ is trivial, so we focus on $a \notin \{0, \star, 1\}$. If $b = \star$, $c = 1$ then $b \wedge_{\mathrm{Ub}} c = \star$. Thus, $a \leq_\ell b = \star = b \wedge_{\mathrm{Ub}} c$. If $b = \star$ and $c \neq 1$, $b \wedge_{\mathrm{Ub}} c = c$. As $a \leq_\ell c$, the property holds. □

4 Partial fuzzy relational equations – the case of Upper boundary algebra

This section addresses the sufficient solvability for a system of partial fuzzy relational equations with operations from the Upper boundary algebra. As we have mentioned above, the system with direct product $\circ_\theta$ was sufficiently solved with help of the lower boundary strategies while the case of the system with the subdirect product $\lhd_\theta$ was not. Therefore, the Upper boundary algebra was designed with the clear goal to be employed in the case of the system with the Bandler-Kohout subproduct $\lhd_\theta$ and therefore, it is the only system that makes sense to be investigated.

Throughout the section, we use $\mathcal{F}(U)$ and $\mathcal{F}^\star(U)$ to denote the sets of all fuzzy sets and the set of all partial fuzzy sets on a given universe U, respectively.

4.1 Fuzzy relational equations with the subdirect product

First, let us recall some essential facts about the standard (non-partial) systems of fuzzy relational equations with the subdirect product that will be helpful for the subsequent analysis. The considered system of fuzzy relational equations is given generally in the following form:

$$A_i \lhd R = B_i, \quad i = 1, \dots, m \tag{6}$$

where $A_i \in \mathcal{F}(X)$, $B_i \in \mathcal{F}(Y)$ are known, and fuzzy relation $R \in \mathcal{F}(X \times Y)$ is unknown. The expanded form of the composition $A \lhd R$ using the Bandler-Kohout subproduct $\lhd$ is expressed as follows:

$$(A \lhd R)(y) = \bigwedge_{x \in X} (A(x) \rightarrow R(x, y)) \ . \tag{7}$$

The interpretation of the system described in (6) is notoriously known however, for the sake of completeness, we feel the duty to recall it. The fuzzy sets A_i and B_i are the given antecedent and consequent fuzzy sets related to a given fuzzy rule base encoding knowledge about certain dependence between inputs $x \in X$ and outputs $y \in Y$ of the modeled system. We seek fuzzy relation R that solves system (6), i.e., a fuzzy relation that if being substituted into the system, the equality remains preserved. As composition is a mathematical model of a fuzzy inference [27, 33], and (7) expresses the formula that determines the output of a fuzzy inference system based on the processed input (observation) given by A, system of equations (6) has a clear interpretation – the preservation of the modus ponens. Indeed, given the antecedents A_i and consequents B_i, we ask, whether if the input fuzzy set A coincides with any of the antecedents A_i, the output inferred by the fuzzy inference coincides with the respective consequent B_i, if the fuzzy rule base is represented by the fuzzy relation R.

Not all systems are solvable. If a system is not solvable, there does not exist any model of a fuzzy rule base with the given antecedents and consequents that would preserve the modus ponens property. If the system is solvable, we ask for the shape of R which is the solution.

The solvability can be easily ensured in advance if we impose the so-called *finitary condition.*

Definition 2 *[28] Let $I = \{1, \ldots, m\}$. We say that fuzzy sets A_i, for $i \in I$ fulfill the* finitary condition *if there exists an $x_i \in X$ such that $A_i(x_i) = 1$ and $A_j(x_i) = 0$, for any $i, j \in I$ such that $i \neq j$.*

Now, we recall a slightly reformulated theorem published in [28] about ensuring the solvability and about the shape of the solution.

Proposition 2 *Let A_i fulfill the finitary condition. Then system (6) is solvable and the following* Mamdani-Assilian model $\check{R}$ *is its solution:*

$$\check{R}(x, y) = \bigvee_{i=1}^{m} (A_i(x) \otimes B_i(y)). \tag{8}$$

Let us note that the shape of the solution is not dependent on the imposed finitary condition but generally, it holds that if system (6) is solvable then $\check{R}$ is its solution, which determines the Mamdani-Assilian model as the primary choice whenever we deal with the inference given by $\triangleleft$.

4.2 Solvability of partial system with the subdirect product

This section considers the system $A_i \triangleleft_{\mathrm{Ub}} R = B_i$ where antecedents, consequents, and consequently even the solution are partial fuzzy sets, i.e., $A_i \in \mathcal{F}^\star(X)$, $B_i \in \mathcal{F}^\star(Y)$, and $R \in \mathcal{F}^\star(X \times Y)$.

We define the partial fuzzy relation $\check{R}_{\mathrm{Ub}} \in \mathcal{F}^\star(X \times Y)$ that relates to the considered system in order to extend the standard Mamdani-Assilian model $\check{R}$ for the considered underlying Upper boundary algebra:

$$\check{R}_{\mathrm{Ub}}(x, y) = \bigvee_{i=1}^{m}{}_{\mathrm{Ub}} \left(A_i(x) \otimes_{\mathrm{Ub}} B_i(y)\right). \tag{9}$$

Furthermore, we recall the notoriously known notion of the *core of a fuzzy set* that, however, can be defined also for partial fuzzy sets.

Definition 3 *[16] Let $A \in \mathcal{F}^\star(X)$. $\mathrm{Core}(A)$ is given by*

$$\mathrm{Core}(A) = \{u \mid A(u) = 1\}.$$

Moreover, we say that A is subnormal *if $\mathrm{Core}(A) = \emptyset$.*

Proposition 3 *For any $i \in \{1, \ldots, m\}$ and for any $y \notin \mathrm{Core}(B_i)$, the following inequality holds*

$$(A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}})(y) \geq_\ell B_i(y) \ . \tag{10}$$

Proof: Let us fix particular i and let us denote $A_i^0 = \{x \mid A_i(x) = 0\}$, and $A_i^\star = \{x \mid A_i(x) = \star\}$. We put $(A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}})(y) = P_1(y) \wedge_{\mathrm{Ub}} P_2(y) \wedge_{\mathrm{Ub}} P_3(y)$ where

$$P_1(y) = \bigwedge\nolimits_{\mathrm{Ub}}{}_{x \in A_i^0} (0 \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y)) \ ,$$
$$P_2(y) = \bigwedge\nolimits_{\mathrm{Ub}}{}_{x \in A_i^\star} (\star \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y)) \ ,$$
$$P_3(y) = \bigwedge\nolimits_{\mathrm{Ub}}{}_{x \notin A_i^0, A_i^\star} (A_i(x) \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y)) \ .$$

Consider P_1. It holds naturally that $0 \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) = 1 \geq_\ell B_i(y)$.

Consider P_2. If $\check{R}_{\mathrm{Ub}}(x, y) = 0$ then necessarily $A_i(x) \otimes_{\mathrm{Ub}} B_i(y) = \star \otimes_{\mathrm{Ub}} B_i(y)$ has to be 0 as well. Thus, also $B_i(y) = 0$. By this fact, $\star \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) \geq_\ell B_i(y)$ holds. Consider $\check{R}_{\mathrm{Ub}}(x, y) \in \{\star, 1\}$. Then it is clear that $\star \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) = 1 \geq_\ell B_i(y)$. Let finally $\check{R}_{\mathrm{Ub}}(x, y) \notin \{0, \star, 1\}$. Then due to the validity of property (3) we get $A_i(x) \otimes_{\mathrm{Ub}} B_i(y) \neq \star$ and thus, $B_i(y) \notin \{\star, 1\}$. Hence, $\star \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) = \check{R}_{\mathrm{Ub}}(x, y) \geq A_i(x) \otimes_{\mathrm{Ub}} B_i(y) = B_i(y)$.

Consider P_3. If $\check{R}_{\mathrm{Ub}}(x, y) = 0$ then $A_i(x) \otimes_{\mathrm{Ub}} B_i(y) = 0$. Since $A_i(x) \notin \{0, \star\}$ we get $B_i(y) \neq \star$. Thus, $A_i(x), B_i(y) \in [0, 1]$ which leads to $A_i(x) \rightarrow (A_i(x) \otimes B_i(y)) \geq B_i(y)$. Hence, $A_i(x) \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) = A_i(x) \rightarrow_{\mathrm{Ub}} (A_i(x) \otimes B_i(y)) \geq_\ell B_i(y)$.

Consider $\check{R}_{\mathrm{Ub}}(x, y) = 1$. Then trivially $A_i(x) \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) = 1 \geq_\ell B_i(y)$.

Consider $\check{R}_{\mathrm{Ub}}(x, y) = \star$ and let $A_i(x) = 1$. Due to the fact $y \notin \mathrm{Core}(B_i)$, $1 \rightarrow_{\mathrm{Ub}} \star = \star \geq_\ell B_i(y)$. Let $A_i(x) \neq 1$. Then we have $A_i(x) \rightarrow_{\mathrm{Ub}} \star = 1 \geq_\ell B_i(y)$.

Finally, consider $\check{R}_{\mathrm{Ub}}(x, y) \notin \{0, \star, 1\}$. Then $A_i(x) \otimes_{\mathrm{Ub}} B_i(y) \neq \star$. This implies $A_i(x) \rightarrow_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x, y) \geq_\ell A_i(x) \rightarrow_{\mathrm{Ub}} (A_i(x) \otimes_{\mathrm{Ub}} B_i(y))$ and with help of property (4), we obtain $A_i(x) \rightarrow_{\mathrm{Ub}} (A_i(x) \otimes_{\mathrm{Ub}} B_i(y)) \geq_\ell B_i(y)$.

Finally, we can conclude the proof using (5). $\square$

Let us try to focus on the opposite inequality. Here, the imposed finitary condition will be helpful.

Proposition 4 *Let A_i fulfill the finitary condition. Then for any $i \in \{1, \ldots, m\}$ and for any $y \in Y$, the following inequality holds*

$$(A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}})(y) \leq_\ell B_i(y) \ .$$

Proof: Let $x_i \in X$ be the point such that $A_i(x_i) = 1$ and $A_j(x_i) = 0$ for any $i \neq j$. Using (2),

$$\begin{aligned}(A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}})(y) &= \bigwedge\nolimits_{x \in X}^{\mathrm{Ub}} \big(A_i(x) \to_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x,y)\big) \leq_\ell A_i(x_i) \to_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}}(x_i,y)\\ &= A_i(x_i) \to_{\mathrm{Ub}} \left((A_i(x_i) \otimes_{\mathrm{Ub}} B_i(y)) \vee_{\mathrm{Ub}} \bigvee\nolimits_{j \neq i}^{\mathrm{Ub}} (A_j(x_i) \otimes_{\mathrm{Ub}} B_j(y)) \right)\\ &= 1 \to_{\mathrm{Ub}} ((1 \otimes_{\mathrm{Ub}} B_i(y)) \vee_{\mathrm{Ub}} 0) = B_i(y).\end{aligned}$$

□

Propositions 3-4 imply the following Corollaries.

Corollary 1 *Let A_i fulfill the finitary condition. Then for any $y \notin \mathrm{Core}(B_i)$, the following holds*

$$(A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}})(y) = B_i(y) \ , \quad i = 1, \ldots, m \ .$$

Corollary 2 *Let A_i fulfill the finitary condition. Furthermore, let $\mathrm{Core}(B_i) = \emptyset$. Then*

$$A_i \lhd_{\mathrm{Ub}} \check{R}_{\mathrm{Ub}} = B_i \ , \quad i = 1, \ldots, m \ .$$

5 Discussion

Corollary 1 establishes the sought equality up to the single exception that is the set of points that belong to the set of core points of the particular consequents. Though this is not the “perfect result”, it is not surprising. Indeed, the solution obtained for the lower boundary algebras and the $\circ_\theta$ inference was also obtained with such an exception. It only restricted the validity of the equality to the values y that did not belong to the set out of the support of the consequents. And as this case mimicks the duality in both, in the system of partial fuzzy relational equations as well as in the partial algebraic strategies, we could expect such a restriction.

Analogously, as the lower boundary algebra strategies with $\circ_\theta$ led to the solvability for the case of the consequents B_i with unlimited supports, Corollary 2 establishes the solvability for the case of subnormal consequents B_i. In both cases, the finitary condition plays an essential role.

One may view the result rather skeptically as subnormal fuzzy sets are not that often used. We may easily oppose such a skepticism. Indeed, there are systems with subnormal fuzzy systems [35]. Such fuzzy sets are frequently studied and play a significant role in the development of distinct theories, such as the inverse problem of data-driven fuzzy modeling [29], the generalization of the calculation formula of possibilistic mean [24], and the five-way approximate decision theory of fuzzy sets [35]. Moreover, the restriction is purely technical. The fuzzy sets employed in the fuzzy systems applied in practice are never continuous, they are discretized. So, it is as easy as to lower the core values from 1, e.g., to 0.999 in order to overcome the technical obstacle and the usual "convex" shape will not be destroyed.

So, the solution is at our disposal. How should we read the results in the context of the previous results published in [16]? Up to our best knowledge and experience, the choice of the bricks used to build the fuzzy inference systems never starts from the inference mechanism. Vice-versa, it is the fuzzy rule base and its model. Indeed, we should be sure whether we use the implicative (also gradual) model [10, 22] or the Mamdani-Assilian [25] first. And then, in the second step, we should pick the inference that is predetermined [34] for the given fuzzy rule base model. And so, this approach should not be judged to be better or worse than the one presented in [16] for the $\circ_\theta$ and the lower boundary algebras. It should be viewed as a complementary result that gives us the possibility to deal with the Mamdani-Assilian model (jointly with $\lhd_{\mathrm{Ub}}$) while so far, we had the possibility to deal only with the implicative model (jointly with $\circ_{\mathrm{D}}$ or $\circ_{\mathrm{Le}}$). And conclusion carries the main message of the investigation.

References

1. Avron, A.: Natural 3-valued logics—characterization and proof theory. The Journal of Symbolic Logic **56**(1), 276–294 (1991)
2. Bandler, W., Kohout, L.J.: Fuzzy relational products and fuzzy implication operators. In: Proc. Int. Workshop on Fuzzy Reasoning Theory and Applications. Queen Mary College, London (1978)
3. Bandler, W., Kohout, L.J.: Semantics of implication operators and fuzzy relational products. International Journal of Man-Machine Studies **12**(1), 89–116 (1980)
4. Barringer, H., Cheng, J.H., Jones, C.B.: A logic covering undefinedness in program proofs. Acta Informatica **21**(3), 251–269 (1984)
5. Běhounek, L., Novák, V.: Towards fuzzy partial logic. In: 2015 IEEE International Symposium on Multiple-Valued Logic (ISMVL). pp. 139–144. IEEE (2015)
6. Běhounek, L., Daňková, M., Dvořák, A.: Free quantification in four-valued and fuzzy bilattice-valued logics. In: International Symposium on Integrated Uncertainty in Knowledge Modelling and Decision Making. pp. 15–26. Springer (2023)
7. Běhounek, L., Daňková, M.: Variable-domain fuzzy sets – Part II: Apparatus. Fuzzy Sets and Systems **38**, 19–43 (2020)
8. Belohlavek, R.: Sup-t-norm and inf-residuum are one type of relational product: Unifying framework and consequences. Fuzzy Sets and Systems **197**, 45–58 (2012)
9. Bochvar, D.A., Bergmann, M.: On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. History and Philosophy of Logic **2**(1-2), 87–112 (1981)
10. Bodenhofer, U., Daňková, M., Štěpnička, M., Novák, V.: A plea for the usefulness of the deductive interpretation of fuzzy rules in engineering applications. In: Proc. 16th IEEE Int. Conf. on Fuzzy Systems. pp. 1567–1572. London (July 2007)
11. Burda, M., Pavliska, V., Murinová, P.: Computation of support and confidence for interval-valued fuzzy association rules. International Journal of Computational Intelligence Systems **13**(1), 1014–1026 (2020)
12. Běhounek, L., Dvořák, A.: Fuzzy relational modalities admitting truth-valueless propositions. Fuzzy Sets and Systems **388**, 38 – 55 (2020)

13. Cao, N., Štěpnička, M.: Compositions of partial fuzzy relations employing the lower estimation approach. In: The 10th International Conference on Knowledge and Systems Engineering (KSE 2018). pp. 146–151. IEEE, HCM, Vietnam (2018)
14. Cao, N., Štěpnička, M., Burda, M., Dolný, A.: Excluding features in fuzzy relational compositions. Expert Systems with Applications **81**, 1–11 (2017)
15. Cao, N., Štěpnička, M.: Fuzzy relational equations employing dragonfly operations. In: 2019 11th International Conference on Knowledge and Systems Engineering (KSE). pp. 1–6. IEEE (2019)
16. Cao, N., Štěpnička, M.: On solvability of systems of partial fuzzy relational equations. Fuzzy Sets and Systems **450**, 87–117 (2022)
17. Cao, N., Štěpnička, M.: Preservation of properties of residuated algebraic structure by structures for the partial fuzzy set theory. International Journal of Approximate Reasoning **154**, 1–26 (2023)
18. Ciucci, D., Dubois, D.: A map of dependencies among three-valued logics. Information Sciences **250**, 162–177 (2013)
19. Dvořák, A., Holčapek, M., Rico, A.: Qualitative integrals on dragonfly algebras. In: 2021 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). pp. 1–6. IEEE (2021)
20. Fitting, M.: A kripke-kleene semantics for logic programs. J. Log. Program. **2**(4), 295–312 (1985)
21. Fitting, M.: Kleene's logic, generalized. Journal of Logic and Computation **1**(6), 797–810 (1991)
22. Jones, H., Charnomordic, B., Dubois, D., Guillaume, S.: Practical inference with systems of gradual implicative rules. IEEE Transactions on Fuzzy Systems **17**(1), 61–78 (2009)
23. Lukasiewicz, J.: On 3-valued logic (1920). McCall, S.(ed.) Polish Logic pp. 16–18 (1967)
24. Luukka, P., Stoklasa, J.: Possibilistic mean based defuzzification for fuzzy expert systems and fuzzy control—lsd for general fuzzy sets. Journal of Computational and Applied Mathematics **441**, 115663 (2024)
25. Mamdani, E.H., Assilian, S.: An experiment in linguistic synthesis with a fuzzy logic controller. Int. J. Man-Mach. Stud. **7**, 1–13 (1975)
26. Novák, V.: Fuzzy type theory with partial functions. Iranian Journal of Fuzzy Systems **16**(2), 1–16 (2019)
27. Pedrycz, W.: Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data. Fuzzy Sets and Systems **16**, 163–175 (1985)
28. Perfilieva, I.: Finitary solvability conditions for systems of fuzzy relation equations. Information Sciences **234**, 29–43 (2013)
29. Perfilieva, I.: Data-driven modeling with fuzzy sets and manifolds. International Journal of Approximate Reasoning **149**, 178–191 (2022)
30. Sobociński, B.: Axiomatization of a partial system of three-value calculus of propositions. The Journal of Computing Systems **1**, 23–55 (1952)
31. Turner, R.: Logics for artificial intelligence. Ellis Horwood Ltd (1984)
32. Štěpnička, M., Cao, N., Běhounek, L., Burda, M., Dolný, A.: Missing values and dragonfly operations in fuzzy relational compositions. International Journal of Approximate Reasoning **113**, 149 – 170 (2019)
33. Štěpnička, M., Jayaram, B.: On the suitability of the Bandler-Kohout subproduct as an inference mechanism. IEEE Transactions on Fuzzy Systems **18**(2), 285–298 (2010)
34. Štěpnička, M., Jayaram, B., Su, Y.: A short note on fuzzy relational inference systems. Fuzzy Sets and Systems **338**, 90–96 (2018)

35. William-West, T.O., Ciucci, D.: Decision-theoretic five-way approximation of fuzzy sets. Information Sciences **572**, 200–222 (2021)