Intf-HybridMem: ω_A-IvE Entropy Analysis via Median and Arithmetic Mean Fuzzy Aggregations

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Abstract. The idea of width-based interval fuzzy entropy is to measure unawareness related to precise membership degrees of elements in the context of interval-valued fuzzy sets. The current work applies previous results regarding the use of well-known aggregation functions endowed with admissible orders, to analyze interval entropies to the *Intf-HybridMem* architecture, which is a fuzzy rule-based system to exploit the access patterns to volatile and non-volatile memories as a recommendation aid in the decision-making process.

Keywords: Interval-valued Entropy · Admissible Interleaving Orders · Hybrid Memory Management · Interval-valued Fuzzy Logic

1 Introduction

The fuzzy entropy measure is conceived as a measure of uncertainty in a fuzzy set, analyzing the disorganized information and fuzziness in the fuzzy set theory [8]. And, the width-based interval fuzzy entropy methodology considers the interval data diameter as a measure of the lack of knowledge and uncertainty in precise membership degrees of elements in Interval-valued Fuzzy Sets (IvFS).

Theoretical results underlying the width-based interval-entropy methods qualify the analysis of the imprecise information related to the width of interval-valued fuzzy values. They also improve the comparison of data extraction from the fuzzy control system inference concerning total orders while still preserving the specialist opinions and main data interpretability.

The interval data comparison is based on the Xu-Yager [17] and admissible interleaving orders [14], the latter considering Decimal Digit Interleaving (DDI) functions and requiring just an injective and increasing function [14].

Following recent contributions, we direct our motivation to the information evaluation in hybrid memory systems. So, the methodology based on the proposal's theoretical constructions named the ωA-IvE entropy methods aims to explore the information of *Intf-HybridMem* approach [11], a system to support the uncertainty modeling in data management for hybrid memory architectures. The methods for the entropy analysis consider input/output data, which are the read/write frequency and access recency, and the output (the migration recommendation). They are validated by evaluations carried out in two proposed aggregation functions related to the above-mentioned admissible orders.

This paper is organized as follows. Sections 2 and 3 recall the relevant concepts necessary to better comprehend our work, including notions of IvFS, their corresponding operators, and aggregation functions within this context. Section 4 provides the study of interval entropies regarding different functions endowed with an admissible order. In Section 5 we have a case study applying our theoretical results in the interval-valued fuzzy inference system called *Intf-HybridMem*. Finally, the last section addresses our concluding remarks and future works.

2 Preliminary on interval-valued fuzzy sets

The main results on interval-valued fuzzy connectives and IvFS are reported below. Let $L([0,1]) = \{[x_1,x_2] | 0 \leq x_1 \leq x_2 \leq 1\}$ be the family of all interval-valued fuzzy values and $L([0,1])$ be the set of closed, non-empty subintervals on $[0,1]$.

According to [18], a fuzzy set A in a nonempty universe U is characterized by its membership function $\mu_A : U \to [0,1]$, and $\mu_A(u)$ interprets the membership degree of an element $u \in U$ in the fuzzy set A. In this sense, a fuzzy set A can be described as a set of ordered pairs: $A = \{(u, \mu_A(u)) \colon u \in U, \mu_A(u) \in [0,1]\}$. The set of all fuzzy sets over U is denoted by $\mathcal{F}(U)$.

An IvFS can be expressed by its interval-valued membership function (IvMF) $\mu_{\mathbb{A}} \colon U \to L([0,1])$ as follows: $\mathbb{A} = \{(x, \mu_{\mathbb{A}}(x)) \colon x \in U \text{ and } \mu_{\mathbb{A}}(x) \in L([0,1])\}$. Let $\mathcal{F}_{IV}(U)$ be the set of all interval-valued fuzzy sets.

The projection functions $l, r \colon L([0,1]) \to [0,1]$ are, respectively, defined by $l([x_1,x_2]) = x_1$ and $r([x_1,x_2]) = x_2$. For $X \in L([0,1])$, $l(X) = \underline{X}$ and $r(X) = \overline{X}$. For n-tuples $\boldsymbol{X} = (X_1,\ldots,X_n) \in L([0,1])^n$, the following subsets are in $[0,1]^n$:

(I) $l(\boldsymbol{X}) = (\underline{X_1},\ldots,\underline{X_n})$ and $r(\boldsymbol{X}) = (\overline{X_1},\ldots,\overline{X_n})$;

(II) $*(\boldsymbol{X}) = [*(\underline{X_1},\ldots,\underline{X_n}), *(\overline{X_1},\ldots,\overline{X_n})]$, when $* \in \{\vee, \wedge\}$.

We know a function $N \colon [0,1] \to [0,1]$ is a fuzzy negation if verifies the border conditions ($N(0) = 1$ and $N(1) = 0$) and the antitonicity property ($x \leq y$ implies $N(x) \geq N(y), \forall x, y \in [0,1]$). N is a strong fuzzy negation (SFN) if $N(N(x)) = x$, $\forall x \in [0,1]$. And, e is an equilibrium point (EP) of N, if $N(e) = e$.

The fuzzy entropy notion is conceived as a methodology to interpret the disorganized information of fuzzy sets (FS). In [8], a function $E \colon \mathcal{F}(U) \to [0,1]$ is called a fuzzy entropy w.r.t. a SFN $N \colon [0,1] \to [0,1]$, which has e as the equilibrium point, when E verifies, $\forall A, B \in \mathcal{F}(U)$ the following properties:

E1: $E(A) = 0$ if and only if A is crisp (non-fuzzy);

E2: $E(A) = 1$ if and only if $A = \{(u, \mu_A(u) = e) \colon u \in U\}$;

E3: $E(A) \leq E(B)$ if A refines B, meaning that: $\mu_A(u_i) \leq \mu_B(u_i)$ when $\mu_B(u_i) \leq e$ and $\mu_A(u_i) \geq \mu_B(u_i)$ when $\mu_B(u_i) \geq e$;

E4: $E(A) = E(A_N)$, where A_N is the complement of A.

2.1 Admissible orders on $\langle L([0,1]), \preceq\rangle$

The use of admissible orders is inspired on relevant contributions, as given by [9, 13], extended to the multivalued logic contexts [5, 10, 19].

A linear order over $L([0,1])$ is a partial order under which every pair of intervals is comparable. A partial order $\preceq$ is an admissible order (A_d-order) if it is linear and refines the Kulisch-Miranker (KM-order) or Product order $\leq$. The degenerate intervals $\mathbf{0}$ and $\mathbf{1}$ correspond, respectively, to the greatest and the smallest elements of $(L([0,1]), \preceq)$ [5].

By [19], let $M_1, M_2\colon [0,1]^2 \to [0,1]$ be aggregation functions (AF) such that $\forall X, Y \in L([0,1])$, the equalities $M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y})$ and $M_2(\underline{X}, \overline{X}) = M_2(\underline{Y}, \overline{Y})$ hold simultaneously only if $X = Y$. The relation, $\preceq_{M_1,M_2}$ on $L([0,1])$,

$$X \preceq_{M_1 M_2} Y \Leftrightarrow \begin{cases} M_1(\underline{X}, \overline{X}) < M_1(\underline{Y}, \overline{Y}) \text{ or} \\ M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y}) \text{ and } M_2(\underline{X}, \overline{X}) \leq M_2(\underline{Y}, \overline{Y}). \end{cases} \tag{1}$$

is an A_d-order. In addition, let $\preceq$ be an admissible order and $X, Y \in L([0,1])$ be intervals such that $\omega(X) = \omega(Y)$, then $X \preceq Y$ implies $X \leq Y$.

Example 1. The $\preceq_{XY}$-order on $L([0,1])$ [17] given by:

$$X \preceq_{XY} Y \Leftrightarrow \begin{cases} \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or} \\ \left(\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}\right) \end{cases} \tag{2}$$

is an A_d-order, and M_1 and M_2 are the sum and difference, respectively.

Let $A\colon L([0,1]) \to [0,1]$ be an injective increasing function such that $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$ called just by admissible interleaving. And , let $A^{(-1)}\colon [0,1] \to L([0,1])$ be its pseudo-inverse defined by $A^{(-1)}(x) = \inf\{X \in L([0,1]) : A(X) \geq x\}$, where the infimum [16] is w.r.t. the admissible order $\preceq_A$. Then, the $\preceq_A$-relation given by $X \preceq_A Y \Leftrightarrow X{=}Y,$ or $A(X) < A(Y)$ is an A_d-order on $L([0,1])$.

In particular, let A be an admissible interleaving named as the decimal-digit interleaving (DDI). For that, the i-th decimal digit of this representation of a real number $x \in [0,1]$ will be denoted by $x^{[i]}$. Note that the same representation can be given to sub-intervals of the unit interval $[0,1]$. Moreover, the infinite decimal expansion of the endpoints of an interval $X = \left[\underline{X}, \overline{X}\right] \subseteq [0,1]$ is indicated by $\left[\underline{X}, \overline{X}\right] = \left[0.\underline{X}^{[1]}\underline{X}^{[2]} \ldots \underline{X}^{[n]}..., \; 0.\overline{X}^{[1]}\overline{X}^{[2]} \ldots \overline{X}^{[n]}...\right]$.

Hence, the interval $[0.24, 0.5]$ is represented by $[0.24\tilde{0}, 0.5\tilde{0}]$. Now, two orderings for interleaving the digits comprising the extremes $\underline{X}$ and $\overline{X}$ of an interval $X{\subseteq}[0,1]$ are related to the same position in their decimal expansions.

Definition 1. *[14] The DDI functions* $\overrightarrow{\mathbf{A}}, \overleftarrow{\mathbf{A}}\colon L([0,1]) \to [0,1]$*, given by*

$$\overrightarrow{\mathbf{A}}(X) = \begin{cases} 0.\underline{X}^{[1]}9\underline{X}^{[2]}9\ldots, & \textit{if } \overline{X} = 1; \\ 0.\underline{X}^{[1]}\overline{X}^{[1]}\underline{X}^{[2]}\overline{X}^{[2]}\ldots, & \textit{otherwise.} \end{cases} \tag{3}$$

$$\overleftarrow{\mathbf{A}}(X) = \begin{cases} 0.9\underline{X}^{[1]}9\underline{X}^{[2]}9\ldots, & \textit{if } \overline{X} = 1; \\ 0.\overline{X}^{[1]}\underline{X}^{[1]}\overline{X}^{[2]}\underline{X}^{[2]}\ldots, & \textit{otherwise.} \end{cases} \tag{4}$$

are admissible orders on $L([0,1])$ called admissible interleaving.

2.2 Negations on $\langle L([0,1]), \preceq\rangle$: representability and duality

Definition 2. *[19] An interval-valued function $\mathbb{N}\colon L([0,1]) \to L([0,1])$ is an interval-valued fuzzy negation (IvFN) if, for all $X, Y \in L([0,1])$, it verifies:*

$\mathbb{N}1$*: $\mathbb{N}(\mathbf{0}) = \mathbf{1}$; and $\mathbb{N}(\mathbf{1}) = \mathbf{0}$;*
$\mathbb{N}2$*: If $X \geq Y$ then $\mathbb{N}(X) \leq \mathbb{N}(Y)$.*

$\mathbb{N}$ is strict if it is continuous w.r.t. the Moore metric, meaning that it is continuous w.r.t. the metric distance $d_M(X,Y) = \max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|)$. And, $\mathbb{N}$ is strictly decreasing if it verifies

$\mathbb{N}3$: If $X > Y$ then $\mathbb{N}(X) < \mathbb{N}(Y)$, $\forall X, Y \in L(0,1)$.

An IvFN $\mathbb{N}$ is called a strong IvFN [12] if $\mathbb{N}$ also satisfies the involutive property:

$\mathbb{N}4$: $\mathbb{N}(\mathbb{N}(X)) = X$, for all $X \in L([0,1])$.

Substituting the KM-order $\leq$ by an A_d-order-order $\preceq$ in Definition 2, $\mathbb{N}$ will be a (strong, strict, frontier) interval-valued fuzzy negation w.r.t. $\preceq$ or just IvFN($\preceq$), as investigated in [1]. Next, we consider the representability of fuzzy connectives, as given in [7, Definition 4.3].

Proposition 1. *[4, Theorem 5.1] Let $N_1, N_2\colon [0,1] \to [0,1]$ be (strict) fuzzy negations such that $N_1 \leq N_2$. The function $\mathbb{N}_{N_1,N_2}\colon L([0,1]) \to L([0,1])$, given as $\mathbb{N}_{N_1,N_2}(X) = [N_1(\overline{X}), N_2(\underline{X})]$ is a (strict) representable IvFN.*

$E^p \in L([0,1])$ is an equilibrium interval for an interval-valued fuzzy negation $\mathbb{N}$ if $\mathbb{N}(E^p) = E^p$. Trivially, $[0,1]$ is an equilibrium interval of representable interval-valued negations. Thus, an equilibrium interval E^p such that $E^p \neq [0,1]$ is called a non-trivial equilibrium interval.

The interval extension of the standard negation $N_S(x) = 1 - x$ w.r.t. the Kulisch-Miranker order is given by: $\mathbb{N}_S(X) = \widehat{N}_S(X) = [1 - \overline{X}, 1 - \underline{X}]$, $\forall X = [\underline{X}, \overline{X}] \in L([0,1])$. Since N_S has a unique equilibrium point, $e_{N_S} = \frac{1}{2}$, then $\forall x \in [0, e_{N_S}]$, $E^p_{\mathbb{N}_S} = [x, 1-x]$ is an equilibrium interval.

Proposition 2. *[14] Let $\mathbb{N}\colon L([0,1]) \to L([0,1])$ be an IvFN w.r.t. an A_d-order-order $\preceq$. If $\mathbb{N}$ has an equilibrium interval on $L([0,1])$, it is unique.*

Let $\mathbb{N}$ be an IvFN. Extending results from Proposition 2, the $\mathbb{N}$-dual operator of $f\colon L([0,1])^n \to L([0,1])$ is given by $f_{\mathbb{N}}(X_1, \ldots X_n) = \mathbb{N}(f(\mathbb{N}(X_1), \ldots, \mathbb{N}(X_n)))$. And, if $\mathbb{N}$ is a strong IvFN, then f is a mutual $\mathbb{N}$-dual function.

Example 2. See [19, Example 3.4], denoting $c = \frac{\underline{X}+\overline{X}}{2}$, $\alpha = \min(c, 1-c)$ and $r = \frac{\overline{X}-\underline{X}}{2}$. The function $\mathbb{N}_{XY}\colon L([0,1]) \to L([0,1])$ given by: $\mathbb{N}_{XY}(X) = [(1-c) - (\alpha - r), (1-c) + (\alpha - r)]$ is a strong IvFN w.r.t. the Xu-Yager's order. Moreover, $\mathbb{N}_{XY}$ has $\left[\frac{1}{4}, \frac{3}{4}\right] \in L^+([0,1])$ as the equilibrium interval and it may also be expressed by:

$$\mathbb{N}_{XY}(X) = \begin{cases} \left[1 - \frac{\overline{X}+3\underline{X}}{2}, 1 - \frac{\overline{X}-\underline{X}}{2}\right], & \text{if } \overline{X} + \underline{X} \leq 1; \\ \left[\frac{\overline{X}-\underline{X}}{2}, 2 - \frac{3\overline{X}+\underline{X}}{2}\right], & \text{otherwise.} \end{cases} \tag{5}$$

2.3 Restricted (dis)similarity functions on $\langle L([0,1]), \preceq\rangle$

We consider the notion of restricted dissimilarity functions (RDF), and their representable expressions on $\langle L([0,1]), \preceq\rangle$, w.r.t. the KM-order, as given in [16, Definition 15]. Additionally, we consider the notion of IvRDF operator based on A_d-order-orders w.r.t. the KM-order, as proposed in [6].

Example 3. [14] Let $\alpha \in (0,1)$ and $d\colon [0,1]^2 \to [0,1]$ be a restricted dissimilarity function and K_α be a weighted mean, $K_\alpha(X) = (1-\alpha)\underline{X} + \alpha\overline{X}$. The function $\mathbb{D}\colon L([0,1]) \to L([0,1])$, defined by

$$\mathbb{D}(X,Y) = [d(K_\alpha(X), K_\alpha(Y)), \max(d(K_\alpha(X), K_\alpha(Y)), \omega(X), \omega(Y))] \quad (6)$$

is an interval-valued restricted dissimilarity function (IvRDF) w.r.t. $\leq$-order.

Example 4. In [6, Corollary 3.7], the mapping $\mathbb{S}\colon L([0,1])^2 \to L([0,1])$ is an IvREF w.r.t. the $\preceq_{XY}$-order, $\forall X, Y \in L([0,1])$, defined by:

$$\mathbb{S}_{XY}(X,Y) = \left[1 - \left|\frac{\underline{X}+\overline{X}}{2} - \frac{\underline{Y}+\overline{Y}}{2}\right| - \frac{\omega(X)+\omega(Y)}{2}, 1 - \left|\frac{\underline{X}+\overline{X}}{2} - \frac{\underline{Y}+\overline{Y}}{2}\right|\right].$$

The methodology to generate IvREF on $\langle L([0,1]), \leq\rangle$ is monotone w.r.t. the width of the intervals: $\omega(X) \leq \omega(Y) \to \mathbb{S}(Y,Y) \preceq \mathbb{S}(X,X), \forall X, Y \in L([0,1])$.

Example 5. Let $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving, and $\preceq_A$ be the order on $L([0,1])$. The function $\mathbb{S}_A\colon L([0,1])^2 \to L([0,1])$ defined by

$$\mathbb{S}_A(X,Y) = [\min(a, 1-\omega(X), 1-\omega(Y)), \max(a, 1-|A(X)-A(Y)|)],$$

where $a = \min\left(\frac{A(X)}{A(Y)}, \frac{A(Y)}{A(X)}\right)$ and $\frac{x}{0} = 1$, is an IvREF w.r.t. $\preceq_A$, as seen in [14].

Example 6. In [14], let $N_e\colon [0,1] \to [0,1]$ be a fuzzy negation given in Eq.(9) considering e as the equilibrium point. The function $\mathbb{R}_{\mathbf{A}}\colon L([0,1])^2 \to L([0,1])$,

$$\mathbb{R}_{\mathbf{A}}(X,Y) = [\max(0, N_e(|\mathbf{A}(X)-\mathbf{A}(Y)|) - \max(\omega(X), \omega(Y))), N_e(|\mathbf{A}(X)-\mathbf{A}(Y)|)]$$

is an interval-valued restricted equivalence function (IvREF) w.r.t. $\preceq_{\mathbf{A}}$-order.

3 Aggregation functions on $\langle L([0,1]), \preceq\rangle$

An interval-valued aggregation function (IvAF) $\mathbb{M}\colon L([0,1])^n \to L([0,1])$ verifies the following conditions, according to [7]:
$\mathbb{M}1$: $\mathbb{M}(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0}$ and $\mathbb{M}(\mathbf{1},\ldots,\mathbf{1}) = \mathbf{1}$; and $\mathbb{M}2$: if $\mathbf{X} = (X_1,\ldots,X_n) \leq \mathbf{Y} = (Y_1,\ldots,Y_n)$, i.e. $X_i \leq Y_i$ for each $i \in \mathbb{N}_n$, then $\mathbb{M}(\mathbf{X}) \leq \mathbb{M}(\mathbf{Y})$.

Extra properties can be demanded for an IvAF:

- $\mathbb{M}3$: $\mathbb{M}(\mathbf{X}_\sigma) = \mathbb{M}(X_{\sigma(1)},...,X_{\sigma(n)}) = \mathbb{M}(X_1,\ldots,X_n) = \mathbb{M}(\mathbf{X}), \forall \mathbf{X} = (X_1,\ldots,X_n) \in L([0,1]))^n$ and for each σ-permutation on $\{1,\ldots,n\}$ (symmetry property);
- $\mathbb{M}4$: If $\mathbb{M}(\mathbf{X}) = \mathbf{0}$ then $\mathbf{X} = (\mathbf{0},\ldots,\mathbf{0})$;

$\mathbb{M}5$: $\mathbb{M}(X, X, \ldots, X) = X$, $\forall X \in L([0,1])$ (idempotency property).

Let $\mathbb{M}\colon L([0,1])^n \to L([0,1])$ be an IvAF. By [3, Def. 3], its left- and right-projections are functions $\underline{\mathbb{M}}, \overline{\mathbb{M}}\colon [0,1]^n \to [0,1]$. And, consider aggregations functions w.r.t admissible orders, i.e. a function $\mathbb{M}\colon L([0,1])^n \to L([0,1])$ satisfying $\mathbb{M}1$ and $\mathbb{M}2$ substituting the Kulisch-Miranker order by an admissible order. Note that for an IvAF w.r.t. an admissible order, satisfying the idempotency property is equivalent to being an averaging function.

Proposition 3. *Let $\mathbb{M}\colon L([0,1])^n \to L([0,1])$ be an IvAF w.r.t. an admissible order $\preceq$. Then, $\mathbb{M}$ is idempotent iff $\mathbb{M}$ is averaging, i.e. for each $X_1, \ldots, X_n \in L([0,1])$, $\min_{\preceq}(X_1, \ldots, X_n) \preceq \mathbb{M}(X_1, \ldots, X_n) \preceq \max_{\preceq}(X_1, \ldots, X_n)$.*

Proof. ($\Rightarrow$) Let $X_i = \min_{\preceq}(X_1, \ldots, X_n)$ and $X_j = \max_{\preceq}(X_1, \ldots, X_n)$. Then, since $\mathbb{M}$ is idempotent, $\min_{\preceq}(X_1, \ldots, X_n) = \mathbb{M}(X_i, \ldots, X_i) \preceq \mathbb{M}(X_1, \ldots, X_n) \preceq \mathbb{M}(X_j, \ldots, X_j) = \max_{\preceq}(X_1, \ldots, X_n)$.
($\Leftarrow$) For $X \in L([0,1])$, $X = \min_{\preceq}(X, \ldots, X) \preceq \mathbb{M}(X, \ldots, X) \preceq \max_{\preceq}(X, \ldots, X)$ $= X$. Therefore, $\mathbb{M}(X, \ldots, X) = X$.

Corollary 1. *Let $\alpha \in [0,1]$. The function $\mathbb{M}_\alpha : L([0,1])^n \to L([0,1])$, given as $\mathbb{M}_\alpha(X, Y) = \mathbf{0}$, if $X = \mathbf{0}$ or $Y = \mathbf{0}$ and otherwise, $\mathbb{M}_\alpha(X, Y) = [\alpha\underline{X} + (1-\alpha)\underline{Y}, \alpha\overline{X} + (1-\alpha)\overline{Y}]$, is an averaging IvAF.*

Proof. Straightforward from Propositions 3 and 7.

Example 7. Let $\alpha \in [0,1]$. By [19, Cor. 6.5], $\mathbb{M}_\alpha\colon L([0,1])^n \to L([0,1])$ given by:

$$\mathbb{M}_\alpha(\mathbf{X}, \mathbf{Y}) = \begin{cases} \mathbf{0}, \text{ if } X = \mathbf{0} \text{ or } Y = \mathbf{0} \\ \left[\sum_{i=1}^n \alpha\underline{X_i} + (1-\alpha)\underline{Y_i}, \sum_{i=1}^n \alpha\overline{X_i} + (1-\alpha)\overline{Y_i}\right], \text{ otherwise} \end{cases} \tag{7}$$

is an averaging IvAF w.r.t. $\preceq_{XY}$-order, verifying $\mathbb{M}3$, $\mathbb{M}4$ and $\mathbb{M}5$ properties. Besides, when $\alpha = 2$, then $\mathbb{M}_{\frac{1}{2}}$ is the generalized arithmetic mean.

Definition 3. *Let $\alpha \in [0,1]$. Then the function $\mathcal{M}_\alpha\colon L([0,1])^n \to L([0,1])$ defined for each $X_1, \ldots, X_n \in L([0,1])$ by*

$$\mathcal{M}_\alpha(X_1, \ldots, X_n) = \begin{cases} X_{(\frac{n+1}{2})}, & \text{if } n \text{ is odd;} \\ \mathbb{M}_\alpha\left(X_{(\frac{n}{2})}, X_{(\frac{n+2}{2})}\right), & \text{if } n \text{ is even;} \end{cases}$$

where $(X_{(1)}, \ldots, X_{(n)})$ is a permutation of $(X_1, \ldots, X_n)$ such that $X_{(1)} \preceq_{XY} X_{(2)} \preceq_{XY} \ldots \preceq_{XY} X_{(n)}$, is called generalized Xu-Yager median.

Proposition 4. *Let $\alpha \in [0,1]$. Then the function $\mathcal{M}_\alpha$ is an idempotent symmetric interval-valued aggregation function w.r.t. $\preceq_{XY}$.*

Proof. We will prove by induction in n that for every $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in L([0,1])$ such that $X_i \preceq_{XY} Y_i$ for each $i \in \mathbb{N}_n$, $X_{(j)} \preceq_{XY} Y_{[j]}$, for each $j \in \mathbb{N}_n$, where $(X_{(1)}, \ldots, X_{(n)})$ and $(Y_{[1]}, \ldots, Y_{[n]})$ are permutations of $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, respectively, such that $X_{(1)} \preceq_{XY} X_{(2)} \preceq_{XY} \ldots \preceq_{XY} X_{(n)}$ and

$Y_{[1]} \preceq_{XY} Y_{[2]} \preceq_{XY} \ldots \preceq_{XY} Y_{[n]}$. If $n = 1$ then $X_{(1)} = X_1 \preceq_{XY} Y_1 = Y_{[1]}$. Suppose that for each $n = k$ it is true. Let $n = k+1$, and $h = [1]$. Then $X_{(1)} \preceq_{XY} X_h \preceq_{XY} Y_h \preceq_{XY} Y_{[1]}$. In addition, $X'_1, \ldots, X'_k, Y'_1, \ldots, Y'_k \in L([0,1])$ such that $X'_j = X_j$ and $Y'_j = Y_j$ for each $j < (1)$ and $X'_j = X_{j+1}$ and $Y'_j = Y_{j+1}$ for each $j \geq (1)$. Then $X'_j \preceq_{XY} Y'_j$ for each $j \in \mathbb{N}_k$. So, by the inductive hypothesis $X'_{(j)} \preceq_{XY} Y'_{[j]}$ for each $j \in \mathbb{N}_k$. Therefore, for each $j \in \mathbb{N}_{k+1}$, $X_{(j)} \preceq_{XY} Y_{[j]}$. So, if n is odd then $\mathcal{M}_\alpha(X_1, \ldots, X_n) = X_{(\frac{n+1}{2})} \preceq_{XY} Y_{[\frac{n+1}{2}]} = \mathcal{M}_\alpha(X_1, \ldots, X_n)$ and if n is even then, by Prop. 7, $\mathcal{M}_{XY}(X_1, \ldots, X_n) = \mathbb{M}_\alpha\left(X_{(\frac{n}{2})}, X_{(\frac{n+2}{2})}\right) \preceq_{XY} \mathbb{M}_\alpha\left(Y_{[\frac{n}{2}]}, Y_{[\frac{n+2}{2}]}\right) = \mathcal{M}_\alpha(Y_1, \ldots, Y_n)$. Thereby, $\mathcal{M}_\alpha$ is increasing w.r.t. $\preceq_{XY}$. Let $X \in L([0,1])$. If n is odd then it is immediate that $\mathcal{M}_\alpha(X, \ldots, X) = X$ and if n is even then, by Corollary 1, $\mathcal{M}_\alpha(X, \ldots, X) = \mathbb{M}_\alpha(X, X) = X$. Hence, $\mathcal{M}_\alpha(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$ and $\mathcal{M}_\alpha(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$. So, $\mathcal{M}_\alpha$ is an idempotent IvAF w.r.t. $\preceq_{XY}$. Besides, as $\mathcal{M}_\alpha(X_1, \ldots, X_n) = \mathcal{M}_\alpha(X_{(1)}, \ldots, X_{(n)})$ then $\mathcal{M}_\alpha$ is also symmetric.

Definition 4. *Let $\alpha \in [0,1]$. Then the function $\mathcal{M}^{\mathbf{A}}_\alpha \colon L([0,1])^n \to L([0,1])$ defined, for each $X_1, \ldots, X_n \in L([0,1])$, by:*

$$\mathcal{M}^{\mathbf{A}}_\alpha(X_1, \ldots, X_n) = \begin{cases} X_{(\frac{n+1}{2})}, & \text{if } n \text{ is odd;} \\ \mathbb{M}_\alpha\left(X_{(\frac{n}{2})}, X_{(\frac{n+2}{2})}\right), & \text{if } n \text{ is even;} \end{cases} \tag{8}$$

where $(X_{(1)}, \ldots, X_{(n)})$ is a permutation of $\mathbf{X}$ such that $X_{(1)} \preceq_{\mathbf{A}} \ldots \preceq_{\mathbf{A}} X_{(n)}$, is called generalized $\mathbf{A}$-median.

Proposition 5. *Let $\alpha \in [0,1]$. Then the function $\mathcal{M}^{\mathbf{A}}_\alpha$ is an idempotent symmetric interval-valued aggregation function w.r.t. $\preceq_{\mathbf{A}}$.*

Proof. Analogous to Proposition 4.

3.1 Negations and Aggregations on $\langle L([0,1]), \preceq_A \rangle$

Main results on fuzzy negations on $\langle L([0,1]), \preceq_A \rangle$ are now reported, since they are used in the ω_A-IvE entropy analysis. More theoretical details, intuitive notions, and illustrative examples based on such concepts are described in [14].

Theorem 1. *[14] Let $A \colon L([0,1]) \to [0,1]$ be an injective and increasing function and $N \colon [0,1] \to [0,1]$ be a strict negation. The function $\mathbb{N}^A \colon L([0,1]) \to L([0,1])$, defined by $\mathbb{N}^A(X) = A^{(-1)}(N(A(X)))$, is an $\langle L([0,1]), \preceq_A \rangle$-negation, named a representable $\langle L([0,1]), \preceq_A \rangle$-negation.*

Proposition 6. *[14] Let A be an admissible interleaving. Whenever $N \colon [0,1] \to [0,1]$ is a strong fuzzy negation, $\mathbb{N}^A$ verifies $\mathbb{N}^A(\mathbb{N}^A(X)) \succeq_A X$.*

Example 8. The $\langle L([0,1]), \preceq_{\mathbf{A}} \rangle$-representable negation generated by the standard negation N_S is given by: $\mathbb{N}^{\mathbf{A}}_S(X) = \mathbf{A}^{(-1)}(N_S(\mathbf{A}(X)))$, $\forall X \in L([0,1])$.

Example 9. [14] Let $N_S\colon [0,1] \to [0,1]$ be the standard negation. The function $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}\colon L([0,1]) \to L([0,1])$ defined by: $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\overrightarrow{\mathbf{A}}(X)))$ is the N_S-interleaving negation on $L([0,1])$ w.r.t. the pairwise $(\preceq_{\overrightarrow{\mathbf{A}}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order.

Example 10. Let $\mathrm{e} \in (0,1)$. Then, $N_{\mathrm{e}}\colon [0,1] \to [0,1]$ given by

$$N_{\mathrm{e}}(x) = \begin{cases} 1 - \frac{(1-\mathrm{e})}{\mathrm{e}}x, & \text{if } x \leq e; \\ \frac{\mathrm{e}}{1-\mathrm{e}}(1-x), & \text{otherwise}; \end{cases} \tag{9}$$

is a strong (strict) fuzzy negation and, it has e as the equilibrium point [15]. The mapping $\mathbb{N}_{\mathrm{e}}^{\mathbf{A}}\colon L([0,1]) \to L([0,1])$, for all $X \in L([0,1])$, defined by

$$\mathbb{N}_{\mathrm{e}}^{\mathbf{A}}(X) = \begin{cases} \overleftarrow{\mathbf{A}}^{(-1)}(N_{\mathrm{e}}(\mathbf{A}(X))), & \text{if } \underline{X} \leq \mathrm{e}, \\ \overrightarrow{\mathbf{A}}^{(-1)}(N_{\mathrm{e}}(\mathbf{A}(X))), & \text{otherwise}; \end{cases} \tag{10}$$

is called the N_e-interleaving negation w.r.t. the pairwise $(\preceq_{\mathbf{A}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order.

Proposition 7. *[14] Let $M\colon [0,1]^n \to [0,1]$ be an aggregation, and $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving. The function $\mathbb{M}^A\colon L([0,1])^n \to L([0,1])$,*

$$\mathbb{M}^A(X_1, \ldots, X_n) = A^{(-1)}\left(M(A(X_1), \ldots, A(X_n))\right) \tag{11}$$

is an IvA w.r.t. $\preceq_A$-order. If M is idempotent, then $\mathbb{M}^A$ is idempotent.

Corollary 2. *Let $M\colon [0,1]^2 \to [0,1]$ be an average function and $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving. If $A(X) \leq A(Y)$, for some $X, Y \in L([0,1])$, then $X \preceq_A \mathbb{M}^A(X,Y) \preceq_A Y$.*

Proof. Straightforward from Proposition 3.

Example 11. Let $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving. When M is the minimum, then $\mathbb{M}^A$ also is the minimum w.r.t. $\preceq_A$. So, by Proposition 7,

$$\mathbb{M}^A(X_1, \ldots, X_n) = A^{(-1)}(M(A(X_1), \ldots, A(X_n))) = A^{(-1)}M_{i=1}^n A(X_i).$$

So, $\mathbb{M}^{\mathbf{A}}$ is an idempotent IvA related to the admissible $\preceq_A$-order.

Corollary 3. *Let M be an idempotent aggregation function and A be an admissible interleaving. The IvAF $\mathbb{M}^A$ w.r.t. $\preceq_A$ is an average function.*

Proof. Straightforward from Propositions 7 and 3.

Definition 5. *Let the mapping $M\colon [0,1]^2 \to [0,1]$ be the arithmetic mean and $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving. So, $\mathcal{M}^A\colon L([0,1])^n \to L([0,1])$ defined, for each $X_1, \ldots, X_n \in L([0,1])$, by:*

$$\mathcal{M}^A(X_1, \ldots, X_n) = \begin{cases} X_{(\frac{n+1}{2})}, & \textit{if } n \textit{ is odd}; \\ \mathbb{M}^A\left(X_{(\frac{n}{2})}, X_{(\frac{n+2}{2})}\right), & \textit{if } n \textit{ is even}; \end{cases}$$

is called generalized interleaving median whenever $(X_{(1)}, \ldots, X_{(n)})$ is a permutation of $(X_1, \ldots, X_n)$ such that $X_{(1)} \preceq_A X_{(2)} \preceq_A \ldots \preceq_A X_{(n)}$.

Proposition 8. *Let $M\colon [0,1]^2 \to [0,1]$ be an average function and $A\colon L([0,1]) \to [0,1]$ be an admissible interleaving. Then the function $\mathcal{M}^A$ is an idempotent symmetric aggregation function w.r.t. $\preceq_A$. So, $\mathcal{M}^A$ is an average function.*

Proof. Analogous to Proposition 4.

4 Width-based interval fuzzy entropy

This section introduces the study of interval entropies generated by IvAF and IvREF w.r.t. admissible the $\preceq$-order.

Definition 6. *[16, Def. 39] Let $\varepsilon \in L([0,1])$ such that $\underline{\varepsilon} > 0$ and $\overline{\varepsilon} < 1$ and take $\leq_L$ as a partial order on $L([0,1])$ such that $\mathbf{0}$ and $\mathbf{1}$ are the least and greatest elements. A function $\mathbb{E}_w : \mathcal{F}_{IV}(U) \to L([0,1])$ is called a width-based interval fuzzy entropy (ω_A-IvE) w.r.t. $\langle \leq_L, \varepsilon\rangle$ if it satisfies the following conditions:*

($\mathbb{E}_w1$) $\mathbb{E}_\omega(\mathbb{A}) = \mathbf{0}$ *iff* $\mathbb{A}$ *is crisp;*
($\mathbb{E}_w2$) $\mathbb{E}_\omega(\widetilde{\varepsilon}) = [1-\omega(\varepsilon), 1]$;
($\mathbb{E}_w3$) $\mathbb{E}_\omega(\mathbb{A}) \leq_L \mathbb{E}_\omega(\mathbb{B})$ *if for all* $u \in U$, $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ *and, either* $\mathbb{A}(u) \leq_L \mathbb{B}(u) \leq_L \varepsilon$ *or* $\varepsilon \leq_L \mathbb{B}(u) \leq_L \mathbb{A}(u)$.

The width-based average functions are taken as aggregation functions. This means that by the action of mean AF, the diameter of the interval input data is preserved into the interval output data in the expression of width-based interval fuzzy entropy (ω_A-IvE) w.r.t. a partial order $\leq$.

Next, let $Av\colon [0,1]^2 \to [0,1]$ be an average fuzzy AF. So, $\widehat{Av}\colon \mathcal{F}_{IV}(U) \to \mathcal{F}(U)$ defined $\forall \mathbb{A} \in \mathcal{F}_{IV}(U)$ and $u \in U$ is given by: $\widehat{Av}(\mathbb{A})(u) = Av(\underline{\mathbb{A}(u)}, \overline{\mathbb{A}(u)})$.

4.1 Width-based interval fuzzy entropy: methodology

Let $\mathbb{A}(u_i) = X_i \in L([0,1])$ be the interval-valued fuzzy value of an element $u_i \in U$ in $\mathbb{A} \in \mathcal{A}_{IV}$, and consider $E^N\colon [0,1] \to [0,1]$ as the fuzzy normal-entropy defined by: $E^N(X_i) = 1 - |2X_i - 1|$. We applied the entropy on [16], taking $\mathbb{A}$ as the average fuzzy arithmetic mean and median. We propose six different methods detailed as follows.

Method 1 Taking $(L([0,1]), \preceq_{XY})$, the interval-valued entropy $\mathbb{E}_\omega\colon \mathcal{F}_{IV}(U) \to L([0,1])$ can be expressed by: $\mathbb{E}_\omega(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_\omega(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_\omega(X_i)$.

Method 2 For all $u_i \in U$, let $(\mathbb{A}(u_i) = X_i \in L([0,1])$, the interval-valued entropy $\mathbb{E}_{\mathbf{A}}\colon \mathcal{F}_{IV}(U) \to L([0,1])$ related to $(L([0,1]), \preceq_{XY})$ is given by: $\mathbb{E}_{\mathbf{A}}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbf{A}}(X_i)$.

Method 3 The interval-valued entropy $\mathbb{E}_{\mathbb{S},\omega}\colon \mathcal{F}_{IV}(U) \to L([0,1])$ is defined by: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbb{S},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbb{S},\omega}(X_i) = \frac{1}{n}\sum_{i=1}^n \mathbb{S}_\omega(X_i, \mathbb{N}(X_i))$, whenever we take $\mathbb{S}_\omega\colon L([0,1])^2 \to L([0,1])$ as the width-based IvREF w.r.t. $\preceq_{XY}$-order, $\mathbb{N}$ as the strong IvFN w.r.t. the $\preceq_{XY}$-order given in Eq. (5).

Method 4 The interval-valued entropy $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}\colon \mathcal{F}_{IV} \to L([0,1])$ is expressed by: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}^N_{\mathbb{S}_{\mathbf{A}},\omega}(X_i)$, whenever we take $\mathbb{S}_{\mathbf{A}}\colon L([0,1])^2 \to L([0,1])$ as the width-based IvREF w.r.t. $\preceq_{\mathbf{A}}$-order, as seen in Example 4.

Method 5 The interval-valued entropy $\mathbb{E}_{\mathbb{R}_{\mathbf{A}},\omega} \colon \mathcal{F}_{IV}(U) \to L([0,1])$, defined by: $\mathbb{E}_{\mathbb{R}_{\mathbf{A}},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{R}_{\mathbf{A}},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{R}_{\mathbf{A}},\omega}(X_i)$, whenever we take $\mathbb{R}_{\mathbf{A}} \colon L([0,1])^2 \to L([0,1])$ as the IvREF w.r.t. $\preceq_{\mathbf{A}}$-order, as in Ex. 6.

Method 6 We also consider the width-based interval fuzzy entropy related to $(L([0,1]), \preceq_{XY})$ introduced in [16, Example 30], and reported as follows: $\mathbb{E}^{p}_{IV}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N_{XY}}(X_i)$.

In the next Section, we apply the six methods consolidated from the theoretical results achieved in the previous sections in the *Intf-HybridMem* [11] architecture.

5 Case study: *Intf-HybridMem* Entropy Analysis via Median and Arithmetic Mean Fuzzy Aggregations

The interval-valued fuzzy inference system supports the uncertainty modeling in data management for hybrid memory architectures, called *Intf-HybridMem* approach [11], exploring decision-making based on the access patterns of temporary storage in volatile memories and, of data persistence in non-volatile memories. It models inaccuracy inherent in input variables, such as read/write frequency and access recency, and also the migration recommendation output. Aiming to improve data management, in a page-level organization, the migration policy verifies each page priority to be switched between memory modules, considering a Rule Base acting on the Fuzzification, Inference, and Defuzzification steps of the IvFL inference system to recommend a correct selection between memory modules. In the *Intf-HybridMem* architecture, four linguistic variables (LV) were defined: RF (reading frequency), WF (writing frequency), and AR (access recency) are the input values, and the output is R (recommendation).

For the current case study, we use ω_A-IvE methods to obtain entropy analysis applied on IvFS considering RF, WF, AR, and R as LV and their linguistic terms (LT) "high" (H), "medium" (M) and "low" (L). The entropy measures consider the six methods from Subsection 4.1.

Table 1: ω_A-IvE Analysis based on Arithmetic Mean Aggregation Function.

M	AR			RF/WF			R		
	E_L	E_M	E_H	E_L	E_M	E_H	E_L	E_M	E_H
1	[0.16 , 0.17]	[0.19 , 0.23]	[0.21 , 0.28]	[0.20 , 0.25]	[0.24 , 0.31]	[0.20 , 0.26]	[0.23 , 0.29]	[0.35 , 0.56]	[0.23 , 0.29]
2	[0.20 , 0.20]	[0.20 , 0.20]	[0.19 , 0.35]	[0.25 , 0.27]	[0.13 , 0.32]	[0.22 , 0.27]	[0.22 , 0.24]	[0.29 , 0.49]	[0.19 , 0.25]
3	[0.09 , 0.17]	[0.12 , 0.23]	[0.14 , 0.28]	[0.13 , 0.25]	[0.16 , 0.31]	[0.13 , 0.26]	[0.14 , 0.29]	[0.28 , 0.56]	[0.15 , 0.29]
4	[0.13 , 0.20]	[0.11 , 0.21]	[0.19 , 0.34]	[0.15 , 0.27]	[0.18 , 0.33]	[0.15 , 0.28]	[0.14 , 0.26]	[0.27 , 0.52]	[0.14 , 0.26]
5	[0.11 , 0.30]	[0.10 , 0.30]	[0.15 , 0.51]	[0.13 , 0.40]	[0.15 , 0.48]	[0.13 , 0.41]	[0.13 , 0.37]	[0.27 , 0.73]	[0.13 , 0.38]
6	[**0.06, 0.17**]	[**0.07, 0.23**]	[**0.04, 0.28**]	[**0.05, 0.25**]	[**0.07, 0.31**]	[**0.05, 0.26**]	[**0.09, 0.29**]	[**0.18, 0.56**]	[**0.09, 0.29**]

See Table 1 summarizing the main results using the methods aggregated by the arithmetic mean (AM) function, discussing its influence on the entropy measure applied to the input/output of the IvFS:

A Firstly, we see that Method 6 provides better results when comparing results applying both order relations, $\preceq_{\mathbf{A}}$- and $\preceq_{XY}$ admissible orders, and a diameter increment of the resulting intervals of most 0.14.

B For the input variables RF/WR, the entropy is greater for Method 1 (M1) in the LT E_M, with the interval $[0.24, 0.31]$. For the output variable R, the entropy is greater for M1 in the LT E_M, with the interval $[0.35, 0.56]$.

Conversely, for the input variable AR, the entropy is smaller for M6 in the LT E_L, with the interval $[0.06, 0.17]$. And, for the output variable R, the entropy is also smaller for M6 in the LT E_L, having the interval $[0.09, 0.29]$.

These observations show consistent information between the ω_A-IvE analysis and the Footprint Of Uncertainty (FOU) using Arithmetic Means IvA.

Table 2: ω_A-IvE Analysis based on Median Aggregation Function.

M	AR			RF/WF			R		
	E_L	E_M	E_H	E_L	E_M	E_H	E_L	E_M	E_H
1	[0.00 , 0.00]	[0.00 , 0.00]	**[0.30, 0.40]**	[0.15 , 0.18]	[0.30 , 0.40]	[0.17 , 0.22]	[0.15 , 0.19]	**[0.31, 0.50]**	[0.17 , 0.21]
2	[0.00 , 0.00]	[0.00 , 0.00]	[0.34 , 0.62]	[0.03, 0.03]	[0.06, 0.09]	**[0.03, 0.04]**	**[0.03, 0.03]**	[0.37 , 0.62]	**[0.03, 0.04]**
3	[0.00 , 0.00]	[0.00 , 0.00]	[0.20 , 0.40]	[0.09 , 0.18]	[0.20 , 0.40]	[0.11 , 0.22]	[0.09 , 0.19]	[0.25 , 0.50]	[0.11 , 0.21]
4	**[0.00, 0.00]**	**[0.00, 0.00]**	[0.34 , 0.59]	**[0.02, 0.02]**	[0.05 , 0.15]	**[0.02, 0.03]**	**[0.02, 0.02]**	[0.34 , 0.59]	**[0.02, 0.03]**
5	[0.00 , 0.00]	[0.00 , 0.00]	[0.24 , 0.90]	[0.00 , 0.08]	[0.00 , 0.93]	[0.00 , 0.10]	[0.00 , 0.08]	[0.24 , 0.90]	[0.00 , 0.10]
6	[0.00 , 0.00]	[0.00 , 0.00]	[0.00 , 0.40]	[0.00 , 0.18]	[0.00 , 0.40]	[0.00 , 0.22]	[0.00 , 0.19]	[0.10 , 0.50]	[0.00 , 0.21]

Observe Table 2 and entropy results applying the median (M) IvA:

A Applying **M** as the generalized median (M) aggregation function, the methods presented more sensible results for the entropy analysis. This is seen by observing the **0** in the first two columns, for the input variable AR, namely the linguistic terms E_L and E_M.

B In general, methods M2 and M4 obtained the lowest entropy in relation to the interval width. For input variables RF, the interval-valued entropy is greater with M1, returning $[0.30, 0.40]$ in the LT E_M.

C For the output variable, the interval-valued entropy is greater in Method 5, for the LT E_M, and smaller in Methods 2 and 4, for the LT E_L and E_H.

So, this demands revision on the IvA selection, since the Median leads to inconsistency analysis by presenting data with zeros for the input variable AR.

6 Conclusion

This article considers two admissible linear orders, the Xu-Yager order and, the order based on the admissible interleaving notion related to DDI functions, considering just one injective and increasing function. Both A_d-orders demand the application of sum and difference operators to compare among elements of an IvFS. And, the algebraic properties defining fuzzy connectives structuring the proposed ω_A-IvE entropy methods are compatible with such admissible orders, $\langle L([0,1]), \preceq_{XY} \rangle$ and $\langle L([0,1]), \preceq_A \rangle$, respectively.

The study of width-based interval entropy applying the ω_A-IvE methodology enables to correlate data information in the modeling of input/output IvFS of the *intf-HybridMem* approach.

Future studies concern the extension of the methods for Atanassov's interval-valued intuitionistic fuzzy sets [2], and the corresponding admissible interleaving generalization.

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