On the paraconsistent companions of involutive fuzzy logics that preserve non-falsity

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Abstract. It is known that most systems of fuzzy logic trivialise in the presence of contradictory information of the type $\{\varphi, \neg\varphi\}$, since with the standard truth-preserving $[0,1]$-valued semantics, there is no evaluation assigning truth-degree 1 to both φ and $\neg\varphi$. In this paper we consider an alternative semantics for some well-known fuzzy logics with an involutive negation (definable or primitive), where an evaluation validates a formula as soon as it gets a non-zero truth-value. This is a paraconsistent semantics, since both φ and $\neg\varphi$ can simultaneously be evaluated with a positive truth-degree without trivialising the reasoning, and it has been called non-falsity preserving semantics by Avron. In this paper we study the properties of this semantics and axiomatise it for the case of several systems of fuzzy logic, among them Łukasiewicz, Nilpotent minimum and Gödel with involution logics.

Keywords: Mathematical fuzzy logic · Non-falsity preserving logics · Involutive negation · IMTL logic · SMTL logic

1 Introduction

Non-classical logics aim at providing models of reasoning in a wide variety of different contexts in which the classical approach might be inadequate or not sufficiently flexible. This is typically the case when the information to reason about is incomplete, imprecise or contradictory.

Paraconsistent logics have been introduced, among other approaches, as systems able to cope with contradictory or inconsistent information and that are expected to extract from it sensible non-trivial inferences, see e.g. [8, 16, 4]. Formally, a logic L is said to be *paraconsistent* with respect to a negation connective $\neg$ when it contains a $\neg$-contradictory but not trivial theory. Assuming that L is (at least) Tarskian, this is equivalent to say that the $\neg$-explosion rule

$$\frac{\varphi \quad \neg\varphi}{\psi}$$

is not valid in Ł.

On the other hand, fuzzy logics are logics of *graded truth* that have been proposed as a suitable tool for reasoning with imprecise information, in particular for reasoning with propositions containing vague predicates. Their main feature is that they allow to interpret formulas in a linearly ordered scale of truth-values, and this is specially suited for representing the gradual aspects of vagueness. In particular, systems of fuzzy logic have been in-depth developed within the frame of mathematical fuzzy logic [7] (MFL). In deductive systems in MFL, mostly with semantics in the real unit interval $[0, 1]$, the usual notion of deduction is defined by requiring the preservation of the truth-value 1 (full *truth-preservation*), which is understood as representing the absolute truth. Namely, generalizing the classical notion of consequence, in these systems a formula follows from a set of premises if every algebraic evaluation that interprets the premises as 1-true also interprets the conclusion as 1-true. This notion of consequence validate the above mentioned explosive rule, so all the fuzzy logics under the truth-preserving paradigm are explosive, and thus they are not paraconsistent.

In the last years, there have been several works studying paraconsistent variants of fuzzy logics (see e.g. [9, 5, 6]), mainly by moving from the (full) truth-preserving paradigm to the *degree-preserving paradigm*, in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is greater or equal than those of the premises, see e.g. [3]. Still, another way of defining paraconsistent variants of a fuzzy logic is put forward in [1], although for the particular case of Łukasiewicz fuzzy logic. In this approach, the notion of consequence at work is the *non-falsity preservation*, according to which a conclusion follows from a set of premises whenever if the premises are non-false, so must be the conclusion. In other words, assuming a $[0, 1]$-valued semantics, this is the case when, for any evaluation, if truth degrees of the premises are above 0, then the truth-degree of the conclusion is so as well. This notion of consequence is weaker than the one in the truth-preserving logics but stronger than the one of degree-preserving logics.

In this paper, we further explore this notion of non-falsity preservation for defining paraconsistent companions of different systems of mathematical fuzzy logic. In more detail, after this introduction, in Section 2 we gather some preliminaries on various systems of t-norm based fuzzy logics. In Section 3 we present basic definitions about variants of these systems specified by logical matrices on MTL-chains with lattice filters as sets of designated values, in particular the non-falsity preserving companions. Then in Section 4 we focus on and axiomatise the paraconsistent non-falsity preserving companions of Involutive MTL logics, while in Section 5 we focus on the companions of SMTL logics with an involutive negation. We conclude with some final remartks in Section 6.

2 Preliminaries

Most well known and studied system of mathematical fuzzy logic are the so-called *t-norm based fuzzy logics*, corresponding to formal many-valued calculi with truth-values in the real unit interval $[0, 1]$ and with a conjunction and an implication interpreted respectively by a (left-) continuous t-norm and its residuum, and thus, including e.g. the well-known Łukasiewicz and Gödel infinitely-valued logics, corresponding to the calculi defined by Łukasiewicz and min t-norms respectively. The most basic t-norm based fuzzy logic is the logic MTL (monoidal t-norm based logic) introduced in [11], whose theorems correspond to the common tautologies of all many-valued calculi defined by a left-continuous t-norm and its residuum [15].

The language of MTL consists of denumerably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge, \&, \to$, and the truth constant $\overline{0}$. Formulas, which will be denoted by lower case greek letters $\varphi, \psi, \chi, \ldots$, are recursively defined from propositional variables, connectives and truth-constant as usual. Further connectives and constants are definable, in particular: $\neg\varphi$ stands for $\varphi \to \overline{0}$ and $\overline{1}$ stands for $\neg\overline{0}$. A Hilbert-style calculus for MTL was introduced in [11] with the following set of axioms:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi \,\&\, \psi \to \varphi$
(A3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$
(A4) $\varphi \wedge \psi \to \varphi$
(A5) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A6) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
(A7b) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9) $\overline{0} \to \varphi$

and whose unique inference rule is *modus ponens*: from φ and $\varphi \to \psi$ derive ψ.

MTL is an algebraizable logic in the sense of Blok and Pigozzi [2] and its equivalent algebraic semantics is given by the variety of MTL-algebras. MTL-algebras can be equivalently introduced as commutative, bounded, integral residuated lattices $\langle A, \wedge, \vee, *, \to, \overline{0}, \overline{1}\rangle$ further satisfying the following prelinearity condition: $(x \to y) \vee (y \to x) = \overline{1}$.

In Table 1 we gather some of the main axiomatic extensions of MTL together with their additional axioms. Of particular interest in this paper is the Involutive MTL logic (IMTL for short), i.e. the axiomatic extension of MTL with the axiom (INV) which enforces the negation $\neg$ to be involutive. IMTL-algebras are just MTL-algebras whose associated negation satisfies the equation $x = \neg\neg x$. The well-known Łukasiewicz logic is the extension of IMTL with the divisibility axiom (Div), and Gödel logic is the extension of MTL with the contraction axiom (Con). Algebras of Ł are usually called MV-algebras and are IMTL-algebras further satisfying the equation $x * (x \to y) = x \wedge y$, while Gödel-algebras are MTL-algebras satisfying the equation $x * y = x \wedge y$.

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	(Inv)
$\neg\varphi \vee ((\varphi \to \varphi \,\&\, \psi) \to \psi)$	(C)
$\varphi \to \varphi \,\&\, \varphi$	(Con)
$\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$	(Div)
$\neg(\varphi \wedge \neg\varphi)$	(PC)
$\neg(\varphi \,\&\, \psi) \vee (\varphi \wedge \psi \to \varphi \,\&\, \psi)$	(WNM)
$\varphi \vee \neg\varphi$	(EM)

Logic	Additional axioms
Strict MTL (SMTL)	(PC)
Involutive MTL (IMTL)	(Inv)
Nilpotent Minimum (NM)	(Inv) and (WNM)
Basic Logic (BL)	(Div)
Strict Basic Logic (SBL)	(Div) and (PC)
Łukasiewicz Logic (Ł)	(Div) and (Inv)
Product Logic (Π)	(Div) and (C)
Gödel Logic (G)	(Con)
Classical Logic (CL)	(EM)

Table 1. Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

Besides enjoying strong completeness as a consequence of their algebraizability, all the logics in Table 1, enjoy completeness with respect to their corresponding classes of algebras on the real-unit interval $[0,1]$, as proved e.g. in [15] for MTL and in [10] for IMTL. Furthermore, Łukasiewicz logic and Gödel logic are even complete w.r.t. a *single* algebra over $[0,1]$, the standard MV-algebra and the standard Gödel algebra respectively, see e.g. [14].

In the following, given a left-continuous t-norm $*$, we will denote by $[\mathbf{0},\mathbf{1}]_*$ the standard MTL-algebra determined by $*$, i.e. $[\mathbf{0},\mathbf{1}]_* = ([0,1], \min, \max, *, \to, 0, 1)$, where $\to$ is the residuum of $*$ and the negation $\neg$ is defined as $\neg x = x \to 0$.

In this paper we will also consider the expansions of SMTL logics (in particular Gödel logic G and Product logic Π) with an additional involutive negation. We will introduce them in Section 5

3 Logics defined by matrices with lattice filters

In the systems of mathematical fuzzy logic considered above, the usual notion of logical consequence has been defined as preservation of the *truth*, represented by the top element of the corresponding algebras. For instance let L be any of the above logics, which we assume to be complete w.r.t. the family $\mathcal{C}_L = \{[\mathbf{0},\mathbf{1}]_* \mid [\mathbf{0},\mathbf{1}]_* \text{ is a L-algebra}\}$ of standard L-algebras. Then the typical notion of logical consequence is the following for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \models^L \varphi \quad \text{if,} \quad \begin{array}{l}\text{for any } [0,1]_* \in \mathcal{C}_* \text{ and any } [0,1]_*\text{-evaluation } e,\\ \text{if } e(\psi) = 1 \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) = 1 \text{ as well.}\end{array}$$

This can be generalised by considering logics defined by logical matrices $M = \langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is a standard L-chain and F is a non-trivial lattice filter of $\mathbf{A}$ i.e. F is either a closed interval $F_a = [a,1]$ with $a \in (0,1]$, or a semi-open interval $F_{(a} = (a,1]$ with $a \in [0,1)$. Considering the filters as sets of designated values, then the logics specified by the matrices $M^*_a = \langle [\mathbf{0},\mathbf{1}]_*, F_a\rangle$ and

$M^*_{(a} = \langle [\mathbf{0},\mathbf{1}]_*, F_{(a}\rangle$ are defined respectively:

$$\Gamma \models_{M^*_a} \varphi \quad \text{if,} \quad \text{for any } [0,1]_* \in \mathcal{C}_* \text{ and any } [0,1]_*\text{-evaluation } e, \text{ if } e(\psi) \geq a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) \geq a \text{ as well.}$$

$$\Gamma \models_{M^*_{(a}} \varphi \quad \text{if,} \quad \text{for any } [0,1]_* \in \mathcal{C}_* \text{ and any } [0,1]_*\text{-evaluation } e, \text{ if } e(\psi) > a \text{ for any } \psi \in \Gamma, \text{ then } e(\varphi) > a \text{ as well.}$$

Now, if $[\mathbf{0},\mathbf{1}]_*$ is a standard IMTL-algebra, with c being the fixpoint of the involutive negation $n(x) = x \to 0$, then it is easy to check that

(i) $\models_{M^*_a}$ is paraconsistent iff $a \leq c$
(ii) $\models_{M^*_{(a}}$ is paraconsistent iff $a < c$

The extreme cases are the 1-preserving logic $\models_{M^*_1} = \models_L$, which is explosive, and the non-falsity preserving logic $\models_{M^*_{(0}}$, which is paraconsistent w.r.t. $\neg$. Observe that the finitary versions of both logics are strongly related.

Lemma 1. *For every pair of formulas φ, ψ the following relation holds:*

$$\varphi \models_{M^*_{(0}} \psi \quad \textit{iff} \quad \neg\psi \models_{M^*_1} \neg\varphi.$$

Proof. It easily follows by observing that, for any $[0,1]_*$-evaluation e, the condition "if $e(\varphi) > 0$ then $e(\psi) > 0$" is equivalent to "if $e(\psi) = 0$ then $e(\varphi) = 0$", and hence, to the condition "if $e(\neg\psi) = 1$ then $e(\neg\varphi) = 1$" as well. □

The same result holds if, instead of an IMTL logic, we consider a SMTL logic expanded with an involutive negation connective $\sim$. Then, we have to replace standard IMTL chains $[\mathbf{0},\mathbf{1}]_*$ by standard SMTL chains with an additional involutive negation function n (interpreting the involutive connective $\sim$), i.e. with chains $[\mathbf{0},\mathbf{1}]^{\mathbf{n}}_* = ([0,1], \min, \max, *, \to, n, 0, 1)$, and then replace in Lemma 1 the (definable) residual negation $\neg$ by the involutive negation $\sim$.

4 Non-falsity preserving companions of IMTL axiomatic extensions

In this section we focus on the characterisation of logics defined by (sets of) matrices of the form $\langle [0,1]_*, F_{(0}\rangle$, for $*$ being an IMTL t-norm. We remind that this means that $*$ is left-continuous and that the residual negation $\neg$, defined as $\neg x = x \to 0 = \sup\{y \in [0,1] \mid x * y = 0\}$, is such that $\neg(\neg x) = x$. Notable examples of IMTL t-norms are Łukasiewicz t-norm (which is continuous) and Nilpotent Minimum t-norm.

Assume L is an axiomatic extension of IMTL, complete with respect to a class of standard algebras $\mathcal{C}_L$, and whose corresponding notion of proof is denoted $\vdash_L$.

Then our aim is to axiomatise the logic defined by the class of matrices $\mathcal{C}^0_L = \{\langle [0,1]_*, F_{(0}\rangle \mid \langle [0,1]_*, F_1\rangle \in \mathcal{C}_L\}$. Note that the logic (semantically) defined by the set of matrices $\mathcal{C}^0_L$ is indeed $\neg$-paraconsistent.

We syntactically define the system nf-L, the non-falsity preserving companion of L, as follows.

Definition 1. *The logic* nf-*L is defined by taking as axioms those of L together with the following inference rules:*

- *Rule of Adjunction: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Reverse Modus Ponens: (MP^r)* $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}, \qquad$ *if* $\quad \vdash_L \varphi \to \psi$

The corresponding notion of proof will be denoted by $\vdash_{\mathsf{nf}\text{-}L}$*.*

The above (MP^r) rule captures the following form of reverse of modus ponens: if $\neg\psi$ is non-false then either $\neg\varphi$ is non-false or $\neg(\varphi \to \psi)$ is non-false. The addition of the disjunct χ both in the premise and in the conclusion of the rule is needed to properly keep track of successive applications of (MP^r), as it will be made clear in Example 1 below.

It is straightforward to check that the logic nf-L is sound w.r.t. the class of matrices $\mathcal{C}_L^0$. Only notice that, on the one hand, if a rule φ/ψ is sound for a matrix $\mathcal{M} = \langle[\mathbf{0},\mathbf{1}]_*, \{1\}\rangle \in \mathcal{C}_L$ then the rule $\neg\psi \vee \chi/\neg\varphi \vee \chi$ is automatically sound for the matrix $\mathcal{M}' = \langle[\mathbf{0},\mathbf{1}]_*, (0,1]\rangle \in \mathcal{C}_L^0$.

In order to show the logic nf-L is complete, we first prove in the following proposition a syntactic counterpart of Lemma 1, relating proofs in L and proofs in nf-L.

Proposition 1. *If* $\psi \vdash_L \varphi$ *then* $\neg\varphi \vdash_{\mathsf{nf}\text{-}L} \neg\psi$*.*

Proof. Suppose $\psi \vdash_L \varphi$, then there is a proof $\langle \Pi_1, \ldots \Pi_r\rangle$, where $\Pi_1 = \psi$, $\Pi_r = \varphi$ and where each Π_i (with $1 < i \leq r$) either:

- is an axiom of L, or
- has been obtained from previous Π_k, Π_j $(k, j < r)$ by the application of the Modus ponens rule (MP).

We show next that we can build a proof for $\neg\psi$ from $\neg\varphi$ in nf-L. We define:
(1) $\Sigma_1 = \neg\Pi_r = \neg\varphi$.
(2) For each $i = 1, ..., r-1$ we do the following: by the iterative construction below, Σ_i will be of the form $\Sigma_i = \Sigma^* \vee \neg\Pi_{r-i+1}$, for some disjunction of formulas Σ^* (in the case $i = 1$ we take $\Sigma^* = \bot$). Then we define:
- If Π_{r-i+1} is an axiom or theorem of L, then $\Sigma_{i+1} = \Sigma_i$.
- If $\Pi_{r-i+1} = \Psi$ has been obtained from previous $\Pi_k = \Phi, \Pi_j = \Phi \to \Psi$ (with $k, j < r$) by the application of Modus ponens rule, then $\Sigma_{i+1} = \Sigma^* \vee \neg\Pi_k \vee \neg\Pi_j$ is obtained from Σ_i by application of (MP^r).
(3) By construction, Σ_r is of the form $\neg\Pi_1 \vee (\bigvee_{i=1,n} \neg\Pi_{k_i})$, where for each k_i, Π_{k_i} is an axiom or theorem of L. Therefore, $\neg\Pi_1 \vee (\bigvee_{i=1,n} \neg\Pi_{k_i}) \to \neg\Pi_1$ is a theorem of L as well. So we define $\Sigma_{r+1} = \Sigma_r \to \Sigma_1$,[1] and thus by using the restricted Modus Ponens rule (r-MP) on Σ_r and the theorem Σ_{r+1}, we finally get $\Sigma_{r+2} = \neg\Pi_1 = \neg\psi$.

[1] Actually, to be formally accurate we should replace the proof step Σ_{r+1} itself by a whole proof of this theorem in L, but for the sake of simplicity we leave it as it is.

As a consequence, after removing possible duplicates in the sequence

$$\langle \Sigma_1, ..., \Sigma_r, \Sigma_{r+1}, \Sigma_{r+2} \rangle,$$

we get a proof of $\neg\psi$ from $\neg\varphi$ in nf-L. □

We provide next a small example of translating a proof in a IMTL logic L into a proof in the logic nf-L, showing the use of the reverse Modus Ponens rule (MP^r).

Example 1. Let L be a axiomatic extension of IMTL, and consider the following derivation

$$\varphi, \psi, \varphi \to (\psi \to \chi) \vdash_{\mathrm{L}} \chi$$

which clearly holds by applying Modus Ponens rule, so the sequence

- $\Pi_1 = \varphi$
- $\Pi_2 = \varphi \to (\psi \to \chi)$
- $\Pi_3 = \psi \to \chi$, by application of (MP) to Π_1 and Π_2
- $\Pi_4 = \psi$
- $\Pi_5 = \chi$, by application of (MP) to Π_4 and Π_3

is a proof of χ from $\{\varphi, \psi, \varphi \to (\psi \to \chi)\}$ in L.

Now let us see how to get a corresponding proof in nf-L for the derivability

$$\neg\chi \vdash_{\mathsf{nf\text{-}L}} \neg\varphi \vee \neg\psi \vee \neg(\varphi \to (\psi \to \chi)).$$

Following the procedure described in the proof of the above proposition, we get that the following sequence

- $\Sigma_1 = \neg\chi$
- $\Sigma_2 = \neg\psi \vee \neg(\psi \to \chi)$, by application of ($\mathrm{MP}^r$) to Σ_1
- $\Sigma_3 = \neg\psi \vee \neg\varphi \vee \neg(\varphi \to (\psi \to \chi))$, by application of ($\mathrm{MP}^r$) to Σ_2

is a proof of $\neg\psi \vee \neg\varphi \vee \neg(\varphi \to (\psi \to \chi))$ from $\neg\chi$ in nf-L. Note that in the second application of (MP^r) we do need its general form with the additional disjunct both in the premise and in the conclusion.

Theorem 1. *The finitary* nf-*L is sound and complete w.r.t. to the class of matrices* $\mathcal{C}_L^0$.

Proof. Soundness is easy and has already been mentioned above. As for completeness, suppose $\{\psi_1, ..., \psi_n\} \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} \in \mathcal{C}_L^0$. This is equivalent to $\neg\varphi \models_{\mathcal{M}'} \neg(\psi_1 \wedge ... \wedge \psi_n)$ for every $\mathcal{M}' \in \mathcal{C}_L$. By completeness of L, there is a proof $\langle \Pi_1, \ldots \Pi_r \rangle$, where $\Pi_1 = \neg\varphi$, $\Pi_r = \neg\psi_1 \vee ... \vee \neg\psi_n$. Now, by the above Proposition 1, there is also a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ in nf-L. Then, if $\Pi_1, ..., \Pi_r$, with $\Pi_1 = \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ and $\Pi_r = \neg\neg\varphi$, is a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$, to get a proof of φ from $\Gamma = \{\psi_1, ..., \psi_n\}$ it is enough to add:

- A previous step $\Pi_0 = \psi_1 \wedge ... \wedge \psi_n$, obtained by $n-1$ applications of the Adjunction rule (Adj) to the premises Γ. Then Π_1 is obtained by applying the (r-MP) rule to Π_0 and the theorem $\psi_1 \wedge ... \wedge \psi_n \to \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$.

- And a final step $\Pi_{r+1} = \varphi$, obtained by applying the (r-MP) rule to Π_r and the theorem $\neg\neg\varphi \to \varphi$. □

As a direct corollary, we have complete axiomatisations of non-falsity preserving companions of Łukasiewicz logic and of Nilpotent Minimum logic. Actually, regarding Łukasiewicz logic, we have mentioned in the introduction that Avron introduces in [1] a paraconsistent extension of the logic T of Anderson and Belnap called FT. This logic, which is presented as "a paraconsistent counterpart of Łukasiewicz Logic $Ł_\infty$" that preserves *non-falsity* ([1, pp. 75] has , in the sense that it takes the semi-open interval $(0, 1]$ as set of designated values, i.e., all values from $[0, 1]$ except the value 0 (falsity). The logic is firstly defined axiomatically over a propositional language with connectives $\wedge$, $\vee$, $\neg$ and $\to_{\mathrm{FT}}$, and then it is proved that FT is semantically characterized by the logic matrix $\langle \mathbf{M_{[0,1]}}, F\rangle$, where $\mathbf{M_{[0,1]}} = ([0, 1], \wedge, \vee, \neg, \to_{\mathrm{FT}}, 0, 1)$ and the filter of designated values is $F = (0, 1]$. In $\mathbf{M_{[0,1]}}$, the operations $\wedge$, $\vee$ and $\neg$ are as in Łukasiewicz logic (i.e. interpreted by min, max and $n(x) = 1 - x$, respectively), but $\to_{\mathrm{FT}}$ is not Łukasiewicz implication (whose truth-function is $x \to y = \min(1, 1 - x + y)$) since it is interpreted by the following truth-function:

$$x \to_{\mathrm{FT}} y = \begin{cases} \max(1 - x, y), & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $\to_{\mathrm{FT}}$ captures the order since it satisfies the relation $x \to_{\mathrm{FT}} y > 0$ iff $x \leq y$. Avron shows nice properties for this logic (semi-relevance, variable-sharing, modus ponens, etc.), but FT is something else than the non-falsity companion of Łukasiewicz logic.

5 The case of expansions of SMTL logic with an involutive negation

In this section we move from IMTL logics to SMTL logics expanded with an involutive negation. A first straightforward observation is that if L is an extension of SMTL, the residual negation $\neg\varphi = \varphi \to 0$ is in fact Gödel negation, whose interpretation in any L-chain is the mapping defined by $\neg x = 1$ if $x = 0$ and $\neg x = 0$ otherwise. Hence $x > 0$ iff $\neg\neg x = 1$. Also, note that the monoidal operation $*$ (strong conjunction) in any SMTL-chain has no zero divisors, i.e. if $x * y = 0$ then either $x = 0$ or $y = 0$.

Let us recall as well that any axiomatic extension L is complete with respect to the class of L-chains, i.e. wrt the set of matrices $C_L = \{\langle \mathbf{A}, F_1\rangle \mid \mathbf{A} \text{ is a L-chain}\}$. As before we will let $C_L^0 = \{\langle \mathbf{A}, F_{(0}\rangle \mid \mathbf{A} \text{ is a L-chain}\}$. Then, the following lemma holds.

Lemma 2. *For any axiomatic extension L of SMTL, the following hold:*
(i) $\varphi \models_{C_L^0} \psi$ *iff* $\neg\neg\varphi \vdash_L \neg\neg\psi$.
(ii) Modus ponens is a valid rule in $\models_{C_L^0}$.

Proof. Property (i) follows from the above observation that any SMTL-evaluation e in a L-chain $\mathbf{A}$ is such that $e(\varphi) > 0$ iff $e(\neg\neg\varphi) = 1$. As for (ii) note that, by definition, $e(\varphi \to \psi) = \sup\{a \in A \mid e(\varphi) * a \leq e(\psi)\}$. So if $e(\varphi \to \psi) > 0$, there exists $a > 0$ such that $e(\varphi) * a \leq e(\psi)$. Now, if $e(\varphi) > 0$ then, since $*$ has no zero-divisors, necessarily $0 < e(\varphi) * a \leq e(\psi)$. □

These are nice properties, however they imply that the falsity-preserving companion of a SMTL logic collapses into classical logic.

Lemma 3. *Let* $\mathbf{A}$ *be a SMTL-chain and let the matrix* $M = (\mathbf{A}, F_{(0)})$. *Then* $\varphi \models_M \psi$ *iff* $\varphi \vdash_{CL} \psi$.

Proof. Since the matrix of classical propositional logic $\langle \mathbf{2}, \{1\}\rangle$, where $\mathbf{2}$ is the 2-element Boolean algebra on $\{0,1\}$, is a submatrix of $M = \langle \mathbf{A}, F_{(0)}\rangle$, then $\varphi \models_M \psi$ implies $\varphi \vdash_{CL} \psi$. Conversely, let the mapping $h : A \to \{0,1\}$ be defined as $h(0) = 0$ and $h(x) = 1$ if $x > 0$. Then it is easy to check that h is a homomorphism of SMTL-algebras. Therefore, if $\varphi \not\models_M \psi$, there is a $\mathbf{A}$-evaluation e such that $e(\varphi) > 0$ and $e(\psi) = 0$. But then, the evaluation $e' = h \circ e$ is a $\mathbf{2}$-evaluation such that $e'(\varphi) = 1$ and $e(\psi) = 0$, hence $\varphi \not\vdash_{CL} \psi$. □

As an immediate consequence of (i) of Lemma 2 and the previous Lemma 3, we get the following corollary, that can be seen as Glivenko-like theorem for the non-falsity preserving companions of SMTL logics.

Corollary 1. *(Glivenko theorem for SMTL) Let L be an axiomatic extension of SMTL. Then* $\varphi \models_{C_L^0} \psi$ *iff* $\neg\neg\varphi \vdash_L \neg\neg\psi$ *iff* $\varphi \vdash_{CL} \psi$.

Therefore, in order to get paraconsistent non-falsity preserving companions of SMTL logics, we turn our attention to expansions of such logics with an involutive negation $\sim$. Indeed, having an involutive negation in the logic makes the corresponding paraconsistent system not collapse into the classical case as we have seen above it happens with axiomatic extensions of SMTL logic. As it can be easily observed, the mapping h defined in the proof of Lemma 3, is no longer an homomorphism in the case the chain $\mathbf{A}$ has an involutive negation in its signature.

For the case of SBL and its extensions Gödel and Product logics, these expansions were defined in [12], and for the more general setting of axiomatic extensions of MTL in [13]. Following the latter, if L is an axiomatic extension of SMTL, then the logic $L_\sim$ is obtained from L by adding the connective $\sim$ to the language of L, together with the following axioms, where $\Delta\varphi := \neg{\sim}\varphi$:

($\sim$1) $\varphi \leftrightarrow {\sim}{\sim}\varphi$
($\sim$2) $\Delta(\varphi \to \psi) \to ({\sim}\psi \to {\sim}\varphi)$
($\sim$3) $\neg\varphi \to {\sim}\varphi$

and the following inference rule: from φ derive $\Delta\varphi$.

In [12] it was proved that $G_\sim$ is complete with respect to a single matrix $C_{G_\sim} = \langle [0,1]_{G_\sim}, F_1\rangle$ over the standard $G_\sim$-chain $[0,1]_{G_\sim} = ([0,1], \min, \max, *_G, \to_G, n, 0, 1)$, where $*_G = \min$, $\to_G$ is Gödel implication and $n(x) = 1 - x$, while $\Pi_\sim$ is complete w.r.t. the set of matrices $C_{\Pi_\sim} = \{\langle [0,1]_{\Pi,n}, F_1\rangle \mid n \text{ is a strong negation in } [0,1]\}$, where $[0,1]_{\Pi,n} = ([0,1], \min, \max, *_\Pi, \to_\Pi, n, 0, 1)$. Similarly, in [13], $\mathrm{SMTL}_\sim$ was proved to be complete w.r.t. the set of matrices $C_{\mathrm{SMTL}_\sim} = \{\langle [0,1]_{*,n}, F_1 \rangle \mid * \text{ is a SMTL t-norm and } n \text{ is a strong negation in } [0,1]\}$.

Now, let $\mathrm{L}_\sim$ be an axiomatic extension of SMTL expanded with an involutive negation, which we assume is complete with respect to a set of matrices $C_{L_\sim}$, and let us consider its corresponding set of matrices with filters $F_{(0}$, $C^0_{L_\sim} = \{\langle [0,1]_{*,n}, F_{(0} \rangle \mid \langle [0,1]_{*,n}, F_1 \rangle \in C_{L_\sim}\}$.

Next lemma is the counterpart of Lemma 2 for expansions of SMTL logic with $\sim$.

Lemma 4. $\varphi \models_{C^0_{L_\sim}} \psi$ *iff* $\neg\neg\varphi \vdash_{L_\sim} \neg\neg\psi$ *(iff* $\vdash_{L_\sim} \neg\neg\varphi \to \neg\neg\psi$*)*

We define now the non-falsity preserving companion of $\mathrm{L}_\sim$ by just adding an inference rule and prove its completeness.

Definition 2. *The logic* nf-$L_\sim$ *is obtained by adding to the axioms and rules of* $L_\sim$ *the following inference rule:*

– *Positive Modus Ponens: (Pos-MP)* $\dfrac{\varphi, \quad \neg\neg\varphi \to \neg\neg\psi}{\psi}$

Finally we show completeness for the non-falsity preserving logic nf-$\mathrm{L}_\sim$.

Theorem 2. nf-$L_\sim$ *is sound and complete wrt the set of matrices* $C^0_{L_\sim}$.

Proof. Soundness is easy. As for completeness, suppose $\psi_1, \ldots, \psi_n \models_{C^0_{L_\sim}} \varphi$. This is equivalent to $\models_{C_{L_\sim}} \neg\neg(\psi_1 \wedge \ldots \wedge \psi_n) \to \neg\neg\varphi$. By completeness of $\mathrm{L}_\sim$, there is a proof $\langle \Pi_1, \ldots \Pi_r \rangle$, where $\Pi_r = \neg\neg(\psi_1 \wedge \ldots \wedge \psi_n) \to \neg\neg\varphi$ and where each Π_i is either an axiom of $\mathrm{L}_\sim$, or has been obtained from previous Π_k, Π_j $(k, j < r)$ and the application of Modus ponens rule. Note that all the Π_i's are theorems of $\mathrm{L}_\sim$. Then, in order to get a proof of ψ in nf-$\mathrm{L}_\sim$, we only need to do the following:

(i) add a previous step $\Pi_0 = \psi_1 \wedge \ldots \wedge \psi_n$ that is obtained from the premises by the Adjunction rule, which is derivable.

(ii) add a final step $\Pi_{r+1} = \varphi$ that is obtained from Π_0 and Π_r by application of the (Pos-MP) rule.

Therefore, the sequence $\Pi_0, \Pi_1, \ldots \Pi_r, \Pi_{r+1}$ is a proof of ψ in the logic nf-L. □

6 Final remarks

In this paper we have explored the definition and axiomatisation of non-falsity preserving companions of two main families of axiomatic extensions of the Monoidal t-norm based fuzzy logic MTL, namely logics L that are extensions of Involutive MTL (IMTL) and of Strict MTL (SMTL). It turns out that if L is an extension of IMTL then nf-L is directly $\neg$-paraconsistent. In contrast, if L is an extension of SMTL then nf-L collapses into classical logic. Therefore, in this latter case we have then considered their expansions $\mathrm{L}_\sim$ with an involutive negation $\sim$, for which the non-falsity preserving companion nf-L becomes properly paraconsistent, and they are axiomatised by adding a single inference rule to $\mathrm{L}_\sim$. In future work we will analyse the expressive power and further properties of these logics from a paraconsistency point of view.

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