A correspondence between methods for ranking elements of a poset and stochastic orderings*

Ignacio Montes[1 [0000−0001−6534−1613]],
Raúl Pérez-Fernández[1 [0000−0001−6966−1681]], and
Bernard De Baets[2 [0000−0002−3876−620X]]

[1] Departamento de Estadística e I.O. y D.M., Universidad de Oviedo, Spain
{imontes,perezfernandez}@uniovi.es
[2] KERMIT, Department of Data Analysis and Mathematical Modelling, Ghent University, Belgium bernard.debaets@ugent.be

Abstract. Stochastic orderings are a commonly used tool in probability theory for comparing random variables or probability distributions. In a recent publication we showed that stochastic orderings are, to some extent, in correspondence with voting procedures. For instance, we demonstrated that the Borda count is equivalent to the comparison of expectations while the Condorcet method is equivalent to statistical preference. This contribution establishes as well a correspondence between stochastic orderings and methods used in the literature for ranking the elements of a poset. Specifically, we show that some well-known methods used in the literature for ranking the elements of a poset, namely the averaged ranking, mutual rank probabilities and the maximal method, are formally equivalent to comparing expectations, statistical preference and multivariate probabilistic preference, respectively.

Keywords: Poset · Stochastic ordering · Averaged ranking · Mutual rank probabilities · Maximal method.

1 Introduction

Stochastic orderings [2, 17] have been widely used in statistics for comparing random variables. Many fields of application, specially that of Economics [13], have successfully used those stochastic orderings in practical problems related to investment decision making and portfolio selection [12]. Among the many different stochastic orderings that can be found in the literature, we may mention the comparison of expectations [16], or even the more general framework of multi-utility representation orderings [10], which are based on comparing univariate distributions; statistical preference [9], or precedence ordering [1], which relies on the joint distribution of pairs of random variables; and probabilistic

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preference [14], which uses the joint distribution of all the random variables to be compared.

From an apparently different perspective, fields as varied as Discrete Mathematics [11] and Chemistry [3, 8] have addressed the problem of selecting the weak order that best represents a given partially ordered set. That weak order is usually built using the information about the linear extensions of the poset that are, roughly speaking, the total orders compatible with the poset. Notable approaches in this domain include the averaged ranking [18], mutual rank probabilities [7] and the maximal method [11].

In this work, we show that those two problems, i.e., comparing random variables and ranking elements of a poset, are actually closely related. In fact, what we show throughout this contribution is that there exists a correspondence between the existing methods in both fields. It is worth noting that the present paper builds upon the line of our previous contribution [15] in which stochastic orderings and voting procedures used in the field of social choice were shown to have many commonalities with each other.

The remainder of the contribution is organized as follows: after introducing some preliminaries and fixing the notation used throughout the contribution in Section 2, Section 3 explains the main methods used to create a ranking on the elements of a poset and Section 4 introduces some of the main stochastic orderings that can be found in the literature. With this background, we devote Section 5 to establish a connection between both frameworks. We conclude the paper in Section 6 with some final remarks and comments.

2 Preliminaries

In the present section we provide some preliminaries on the theory of partially ordered sets [5]. A partially ordered set, or *poset* for short, is a pair $(P, \leq_P)$ formed by a set P and a reflexive, antisymmetric and transitive relation $\leq_P$ on P. For the sake of simplicity, whenever no confusion is possible we will skip the subindex P for the relation $\leq_P$. Two elements $x, y \in P$ are called *comparable* when it either holds that $x \leq y$ or $y \leq x$; otherwise they are called *incomparable*, denoted by $x \parallel y$. If $x \leq y$ and $x \neq y$, we write $x < y$. Any subset $P' \subseteq P$ defines a poset $(P', \leq_{P'})$, where $\leq_{P'}$ is the restriction of $\leq$ to P'. The *dual* of a poset $(P, \leq)$ is the poset $(P, \geq)$ such that $x \geq y$ if and only $y \leq x$. A poset and its dual are used interchangeably.

In a poset $(P, \leq)$, an element $x \in P$ for which there does not exist another element $y \in P$ such that $x \leq y$ is called *maximal*, whereas an element $x \in P$ for which there does not exist another element $y \in P$ such that $y \leq x$ is called *minimal*. If there exists one unique maximal element it is called the *top*, whereas if there exists one unique minimal element it is called the *bottom*. A poset may only admit one top and one bottom and, in case both of them exist, the poset is called *bounded*.

Given a poset $(P, \leq)$, an element $x \in P$ is said to *be covered by* a different element $y \in P$, denoted by $x \lessdot y$, if it holds that $x < y$ and there does not exist another $z \in P \backslash \{x, y\}$ such that $x < z < y$. The covering relation $\lessdot$ characterizes the order relation $\leq$ and is typically used for graphically representing a poset, resulting in the so-called Hasse diagram.

Example 1. Consider the poset $(P, \leq)$ where $P = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\leq$ is determined by the covering relation

$$\lessdot = \{(x_3, x_1), (x_4, x_1), (x_5, x_4), (x_6, x_2), (x_6, x_5)\}\,.$$

The Hasse diagram is shown in Figure 1. There, a line between two elements means that the element situated below is covered by the element situated above. This diagram allows to understand the structure of the poset and, for example, easily shows that x_1, x_2 are two maximal elements, while x_3 and x_6 are two minimal elements.

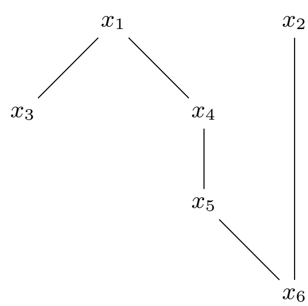

Fig. 1. Hasse diagram for representing the poset in Example 1.

A *chain* is a poset in which all the elements are comparable. A poset $(P, \leq')$ is called an *extension* of $(P, \leq)$ if $x \leq y$ implies $x \leq' y$. A *linear extension* of a poset is an extension of the poset that is a chain. The set of all linear extensions of a poset $(P, \leq)$ is denoted by $E_{\leq}$. As an example, Table 1 shows all nineteen linear extensions of the poset given in Figure 1.

A *weak order or total preorder is a pair* $(P, \precsim)$ *formed by a set* P *and a reflexive, complete and transitive binary relation* $\precsim$ *on* P*. Any weak order relation* $\precsim$ *may be partitioned into two binary relations:*

(i) the strict part $\prec$*: defined by* $x \prec y$ *if* $x \precsim y$ *and* $y \not\precsim x$*. The relation* $(\prec \cup =)$*, simply denoted by* $\preceq$ *if no confusion can occur, is reflexive, transitive and antisymmetric, i.e.,* $\preceq$ *is an order relation.*

(ii) the symmetric part $\sim$*: defined by* $x \sim y$ *if* $x \precsim y$ *and* $y \precsim x$*. The relation* $\sim$ *is reflexive, transitive and symmetric, i.e.,* $\sim$ *is an equivalence relation.*

A weak order $(P, \precsim)$ *is called a* complete extension *of a poset* $(P, \leq)$ *if* $x \leq y$ *implies* $x \precsim y$*. Obviously, any linear extension of a poset is a complete extension.*

Table 1. Linear extensions of the poset represented in Figure 1.

e_1: $x_3 < x_6 < x_2 < x_5 < x_4 < x_1$	e_2: $x_3 < x_6 < x_5 < x_2 < x_4 < x_1$
e_3: $x_3 < x_6 < x_5 < x_4 < x_1 < x_2$	e_4: $x_3 < x_6 < x_5 < x_4 < x_2 < x_1$
e_5: $x_6 < x_2 < x_3 < x_5 < x_4 < x_1$	e_6: $x_6 < x_2 < x_5 < x_3 < x_4 < x_1$
e_7: $x_6 < x_2 < x_5 < x_4 < x_3 < x_1$	e_8: $x_6 < x_3 < x_2 < x_5 < x_4 < x_1$
e_9: $x_6 < x_3 < x_5 < x_2 < x_4 < x_1$	e_{10}: $x_6 < x_3 < x_5 < x_4 < x_1 < x_2$
e_{11}: $x_6 < x_3 < x_5 < x_4 < x_2 < x_1$	e_{12}: $x_6 < x_5 < x_2 < x_3 < x_4 < x_1$
e_{13}: $x_6 < x_5 < x_2 < x_4 < x_3 < x_1$	e_{14}: $x_6 < x_5 < x_3 < x_2 < x_4 < x_1$
e_{15}: $x_6 < x_5 < x_3 < x_4 < x_1 < x_2$	e_{16}: $x_6 < x_5 < x_3 < x_4 < x_2 < x_1$
e_{17}: $x_6 < x_5 < x_4 < x_2 < x_3 < x_1$	e_{18}: $x_6 < x_5 < x_4 < x_3 < x_1 < x_2$
e_{19}: $x_6 < x_5 < x_4 < x_3 < x_2 < x_1$	

3 Ranking elements of a poset

The problem of selecting the most natural complete extension of a given poset has attracted the interest of the scientific community for decades (see, e.g., [4, 11]). Given a poset $(P, \leq)$*, three prominent methods for selecting a complete extension* $(P, \precsim)$ *of* $(P, \leq)$ *are* Averaged rankings *[18],* Mutual rank probabilities *[7] and* Maximal method *[11].*

3.1 Averaged rankings

Next we consider the linear extensions $e \in E_{\leq}$ *and the order relation they determine, denoted by* $\leq_e$*. We define the position* $\mathrm{Pos}_e(x)$ *of an element* x *in a linear extension* $e = (P, \leq_e)$ *as the number of elements in* P *that are ranked at a position that is better (or equal) than that of* x *in the chain determined by* $\leq_e$*, i.e.,* $\mathrm{Pos}_e(x) = |\{z \in P \mid x \leq_e z\}|$*, where* $|A|$ *denotes the cardinality of a finite set* A*. The average position* $\mathrm{av}(x)$ *of* x *in a poset* $(P, \leq)$ *is defined as*

$$\mathrm{av}(x) = \frac{1}{|E_{\leq}|} \sum_{e \in E_{\leq}} \mathrm{Pos}_e(x)\,.$$

The averaged ranking approach [3, 18] selects as the complete extension $(P, \precsim_{\mathrm{av}})$ *of* $(P, \leq)$ *the weak order defined as* $x \precsim_{\mathrm{av}} y$ *if* $\mathrm{av}(x) \leq \mathrm{av}(y)$*. We also consider the notation* $x \prec_{\mathrm{av}} y$ *when* $\mathrm{av}(x) < \mathrm{av}(y)$ *and* $x \sim_{\mathrm{av}} y$ *when* $\mathrm{av}(x) = \mathrm{av}(y)$*.*

Example 2. Consider the poset defined in Figure 1 and the linear extensions of Table 1. The following table enumerates the positions of each one of the elements:

Position	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*
x_1	15	4	0	0	0	0
x_2	4	4	4	4	3	0
x_3	0	3	4	4	4	4
x_4	0	8	8	3	0	0
x_5	0	0	3	8	8	0
x_6	0	0	0	0	4	15

We compute the averaged ranking of each element, obtaining the following values:

$$\mathrm{av}(x_1) = \tfrac{23}{19}\,, \qquad \mathrm{av}(x_2) = \tfrac{55}{19}\,, \qquad \mathrm{av}(x_3) = \tfrac{97}{19}\,.$$
$$\mathrm{av}(x_4) = \tfrac{71}{19}\,, \qquad \mathrm{av}(x_5) = \tfrac{100}{19}\,, \qquad \mathrm{av}(x_6) = \tfrac{129}{19}\,.$$

Thus, we conclude that:

$$x_6 \prec_{\mathrm{av}} x_5 \prec_{\mathrm{av}} x_3 \prec_{\mathrm{av}} x_4 \prec_{\mathrm{av}} x_2 \prec_{\mathrm{av}} x_1\,.$$

It is worth mentioning that the weak order obtained in this example is a total order. However, this is not always the case for the averaged ranking because two elements may have the same average position, as can be seen in the forthcoming Example 4.

3.2 Mutual rank probabilities

Given a poset $(P, \leq)$, we define the mutual rank probability $p_{y<x}$ of $x \in P$ over a different element $y \in P$ as the proportion of linear extensions of $(P, \leq)$ in which $y < x$, that is, $p_{y<x} = \frac{|\{e \in E_{\leq} \mid y \leq_e x\}|}{|E_{\leq}|}$. The method of mutual rank probabilities [8] selects as the complete extension $(P, \precsim_{\mathrm{mr}})$ of $(P, \leq)$ the binary relation defined as $x \precsim_{\mathrm{mr}} y$ if $p_{y<x} \leq \frac{1}{2}$. Note that the relation $\precsim_{\mathrm{mr}}$ is not necessarily transitive and, therefore, its transitive closure $\precsim'_{\mathrm{mr}}$ needs to be considered if the aim is to obtain a complete extension of $(P, \leq)$. Again, we use the notation $x \prec_{\mathrm{mr}} y$ or $x \prec'_{\mathrm{mr}} y$ and $x \sim_{\mathrm{mr}} y$ or $x \sim'_{\mathrm{mr}} y$ for the strict and symmetric parts of $\precsim_{\mathrm{mr}}$ and $\precsim'_{\mathrm{mr}}$, respectively.

Example 3. Consider the poset defined in Figure 1 and the linear extensions of Table 1. The mutual rank probabilities $p_{y<x}$, where x is the element in the row and y is the element in the column, are as follows:

$p_{y<x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	$\frac{15}{19}$	1	1	1	1
x_2	$\frac{4}{19}$	0	$\frac{13}{19}$	$\frac{9}{19}$	$\frac{14}{19}$	1
x_3	0	$\frac{6}{19}$	0	$\frac{5}{19}$	$\frac{10}{19}$	$\frac{15}{19}$
x_4	0	$\frac{10}{19}$	$\frac{14}{19}$	0	1	1
x_5	0	$\frac{5}{19}$	$\frac{9}{19}$	0	0	1
x_6	0	0	$\frac{4}{19}$	0	0	0

Thus, we conclude that:

$$x_6 \prec_{\mathrm{mr}} x_5 \prec_{\mathrm{mr}} x_3 \prec_{\mathrm{mr}} x_2 \prec_{\mathrm{mr}} x_4 \prec_{\mathrm{mr}} x_1\,.$$

Note that $\precsim_{\mathrm{mr}}$ already is transitive, but as will be shown in the following example, this is not always the case.

Example 4. *As mentioned above, the main problem of this method is its lack of transitivity [7], as can be seen in the poset defined in Figure 2. In such example, already presented in [7],* x_1, x_2, x_3 *are preferred to* $x_4, x_5, x_6, x_7, x_8, x_9$ *(*$x_i \prec_{\mathrm{mr}} x_j$*, with* $i \in \{4,5,6,7,8,9\}$ *and* $j \in \{1,2,3\}$*), and* $x_1, x_2, x_3, x_4, x_5, x_6$ *are preferred to* x_7, x_8, x_9 *(*$x_i \prec_{\mathrm{mr}} x_j$*, with* $i \in \{7,8,9\}$ *and* $j \in \{1,2,3,4,5,6\}$*). However, there are three cycles: the first one is formed by* x_1, x_2, x_3 *(*$x_1 \prec_{\mathrm{mr}} x_2 \prec_{\mathrm{mr}} x_3 \prec_{\mathrm{mr}} x_1$*), the second one is formed by* x_4, x_5, x_6 *(*$x_4 \prec_{\mathrm{mr}} x_5 \prec_{\mathrm{mr}} x_6 \prec_{\mathrm{mr}} x_4$*) and the third one is formed by* x_7, x_8, x_9 *(*$x_7 \prec_{\mathrm{mr}} x_8 \prec_{\mathrm{mr}} x_9 \prec_{\mathrm{mr}} x_7$*). Therefore, the method of mutual rank probabilities does not rank all the elements in the poset, it does not even provide a transitive relation.*

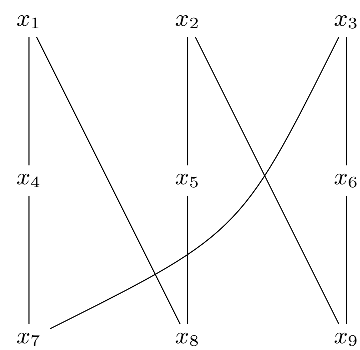

Fig. 2. Hasse diagram of a poset with cyclical statistical preference relation.

The transitive closure of $\precsim_{\mathrm{mr}}$ *leads to:*

$$x_9 \sim'_{\mathrm{mr}} x_8 \sim'_{\mathrm{mr}} x_7 \prec'_{\mathrm{mr}} x_6 \sim'_{\mathrm{mr}} x_5 \sim'_{\mathrm{mr}} x_4 \prec'_{\mathrm{mr}} x_3 \sim'_{\mathrm{mr}} x_2 \sim'_{\mathrm{mr}} x_1 \,.$$

Additionally, this example also shows that the weak order provided by the averaged ranking is not a total order in general, even though it is a total pre-order, since straightforward computations show that

$$x_9 \sim_{\mathrm{av}} x_8 \sim_{\mathrm{av}} x_7 \prec_{\mathrm{av}} x_6 \sim_{\mathrm{av}} x_5 \sim_{\mathrm{av}} x_4 \prec_{\mathrm{av}} x_3 \sim_{\mathrm{av}} x_2 \sim_{\mathrm{av}} x_1 \,.$$

3.3 Maximal method

Given a poset $(P, \leq)$*, the maximal method [11] proceeds in an iterative manner as follows. Let* M_1 *be the set of maximal elements of* $\leq$ *in* P*,* M_2 *be the set of maximal elements of* $\leq$ *restricted to* $P \backslash M_1$ *and, iterating while possible, let* M_i *be the set of maximal elements in the restriction of* $\leq$ *to* $P \backslash \cup_{j=1}^{i-1} M_j$*. The maximal method selects as the complete extension* $(P, \precsim_{\max})$ *of* $(P, \leq)$ *the weak order defined as* $x \precsim_{\max} y$ *if* $x \in M_i$ *and* $y \in M_j$ *for* $i \geq j$*. The notation* $x \prec_{\max} y$ *is used when* $i > j$ *and* $x \sim_{\max} y$ *when* $i = j$*.*

Example 5. Consider the poset defined in Figure 1. It follows that $M_1 = \{x_1, x_2\}$, $M_2 = \{x_3, x_4\}$, $M_3 = \{x_5\}$ *and* $M_4 = \{x_6\}$. *Thus, we conclude that:*

$$x_6 \prec_{\max} x_5 \prec_{\max} x_4 \sim_{\max} x_3 \prec_{\max} x_2 \sim_{\max} x_1 \,.$$

Note that one could define an analogous method based on the minimal elements.

4 Stochastic orderings

Stochastic orderings [2, 17] are methods for the comparison of random variables. In the following, we present three popular options that have been extensively used in the literature: expected value *[16],* statistical preference *[9] and* probabilistic preference *[14].*

In this section we use X, Y, Z *or* $X_1, X_2, X_3, \ldots$ *for denoting random variables defined on the same probability space, where* π *is the probability measure. As well, we use* $X \succsim Y$ *for denoting the preference relation determined by a stochastic ordering (*X *is preferred to* Y*),* $X \succ Y$ *to denote its strict preference relation (meaning that* X *is strictly preferred to* Y*) and* $X \sim Y$ *for denoting its associated indifference relation (meaning that* X *and* Y *are equally preferred).*

4.1 Expected value

Possibly, the most well-known stochastic ordering is that of expected value [16]: a random variable X *is said to be preferred to a random variable* Y *with respect to expected value, denoted by* $X \succsim_{\text{EV}} Y$, *if it holds that* $E(X) \geq E(Y)$.

Example 6. Consider the example provided in [9] in which the numbers from 1 *to* 18 *are distributed over the faces of three dice as follows:* $D_1 = \{1, 3, 4, 15, 16, 17\}$, $D_2 = \{2, 10, 11, 12, 13, 14\}$ *and* $D_3 = \{5, 6, 7, 8, 9, 18\}$. *The objective is to compare the associated random variables* X_1, X_2 *and* X_3, *assuming a uniform distribution over* D_1, D_2 *and* D_3, *respectively. It holds that* $E(X_1) = \frac{28}{3}$, $E(X_2) = \frac{31}{3}$ *and* $E(X_3) = \frac{53}{6}$. *Thus,* $X_2 \succ_{\text{EV}} X_1 \succ_{\text{EV}} X_3$.

4.2 Statistical preference

Another usual stochastic ordering is that of statistical preference [9]. This stochastic ordering is based on the notion of pairwise winning probabilities, formalized by means of a probabilistic relation Q *that measures the degree to which a random variable* X *is greater than another random variable* Y. *The pairwise winning probability of a random variable* X *over another random variable* Y, *denoted by* $Q(X, Y)$, *is defined as follows:*

$$Q(X, Y) = \pi(X > Y) + \frac{1}{2}\pi(X = Y)\,.$$

X is said to be statistically preferred *to Y, denoted by $X \succsim_{\mathrm{SP}} Y$, if $Q(X,Y) \geq Q(Y,X)$. Note that, since $Q(X,Y)+Q(Y,X)=1$, $Q(X,Y) \geq Q(Y,X)$ is equivalent to $Q(X,Y) \geq \frac{1}{2}$. The main drawback of this stochastic ordering is its lack of transitivity [6], as one may find three random variables X, Y and Z such that $X \succ_{\mathrm{SP}} Y$, $Y \succ_{\mathrm{SP}} Z$ and $Z \succ_{\mathrm{SP}} X$.*

Example 7. Continue with the random variables in Example 6. It holds that $Q(X_1,X_2)=\frac{5}{9}$, $Q(X_2,X_3)=\frac{25}{36}$ and $Q(X_3,X_1)=\frac{7}{12}$. It follows that $X_1 \succ_{\mathrm{SP}} X_2$, $X_2 \succ_{\mathrm{SP}} X_3$ and $X_3 \succ_{\mathrm{SP}} X_1$.

4.3 Probabilistic preference

As an alternative to statistical preference that circumvents its lack of transitivity, the notion of probabilistic preference was recently introduced [14]. This stochastic ordering is based on the notion of multivariate winning probability of a random variable X in a finite set of distinct random variables $\mathcal{A}$, which represents the probability of X being the preferred random variable in $\mathcal{A}$, defined as follows:

$$\Pi_{\mathcal{A}}(X) = \sum_{\mathcal{Y} \subseteq \mathcal{A} \setminus \{X\}} \frac{1}{1+|\mathcal{Y}|} \pi\left((\forall Z \in \mathcal{Y}) \big(\forall W \in \mathcal{A} \setminus (\{X\} \cup \mathcal{Y})\big) \big(X = Z > W\big) \right) .$$

It follows that $\sum_{X \in \mathcal{A}} \Pi_{\mathcal{A}}(X) = 1$. Interestingly, in case $\mathcal{A} = \{X,Y\}$, the multivariate winning probability of X in $\{X,Y\}$ reduces to the pairwise winning probability of X over Y, i.e., $\Pi_{\{X,Y\}}(X) = Q(X,Y)$. This is to be understood as follows: if only two random variables are being compared, then both $\Pi_{\{X,Y\}}(X)$ and $Q(X,Y)$ may be used interchangeably; however, if more than two random variables are being compared, then only $\Pi_{\mathcal{A}}(X)$ should be used as it fairly considers all relationships within $\mathcal{A}$.

When $\pi(X=Y)=0$ for any pair of random variables in $\mathcal{A}$, the multivariate winning probability simplifies to:

$$\Pi_{\mathcal{A}}(X) = \pi\big((\forall Y \in \mathcal{A} \setminus \{X\})(X > Y) \big) ,$$

that is, $\Pi_{\mathcal{A}}(X)$ is the probability of X taking a larger value than all the other random variables in $\mathcal{A}$.

Since it is possible that many random variables $X \in \mathcal{A}$ will be such that $\Pi_{\mathcal{A}}(X)=0$, we could consider the following procedure for obtaining a weak order on $\mathcal{A}$. We define $\mathcal{A}_1$ to be the set of random variables $X \in \mathcal{A}$ such that $\Pi_{\mathcal{A}}(X) > 0$, $\mathcal{A}_2$ to be the set of random variables $X \in \mathcal{A} \setminus \mathcal{A}_1$ such that $\Pi_{\mathcal{A} \setminus \mathcal{A}_1}(X) > 0$, and, iterating while possible, $\mathcal{A}_i$ to be the set of random variables $X \in \mathcal{A} \setminus \cup_{j=1}^{i-1} \mathcal{A}_j$ such that $\Pi_{\mathcal{A} \setminus \cup_{j=1}^{i-1} \mathcal{A}_j}(X) > 0$. A random variable X is said to be probabilistically preferred *to another random variable Y (given a finite set of distinct random variables $\mathcal{A}$ with $X,Y \in \mathcal{A}$), denoted by $X \succsim_{\mathrm{PP}} Y$, if it either holds that there exists i such that $X,Y \in \mathcal{A}_i$ and $\Pi_{\mathcal{A} \setminus \cup_{j=1}^{i-1} \mathcal{A}_j}(X) \geq \Pi_{\mathcal{A} \setminus \cup_{j=1}^{i-1} \mathcal{A}_j}(Y)$ or $X \in \mathcal{A}_i$ and $Y \in \mathcal{A}_j$ for $i > j$.*

Example 8. Continue with the random variables in Example 6 and consider the set of random variables $\mathcal{A} = \{X_1, X_2, X_3\}$. *It holds that* $\Pi_{\mathcal{A}}(X_1) = 0.4167$, $\Pi_{\mathcal{A}}(X_2) = 0.3472$ *and* $\Pi_{\mathcal{A}}(X_3) = 0.2361$. *Since* $\mathcal{A}_1 = \{X_1, X_2, X_3\}$, *it holds that* $X_1 \succ_{\text{PP}} X_2 \succ_{\text{PP}} X_3$.

5 A correspondence between methods for ranking elements of a poset and stochastic orderings

Given a poset $(P, \leq)$, *we may define the probability space* $(E_{\leq}, \mathcal{P}(E_{\leq}), \mathcal{U})$, *where* $\mathcal{U}$ *denotes the (discrete) uniform distribution on a finite set. Note that any element* $a \in P$ *defines a random variable* $X_a : E_{\leq} \to \{1, 2, \ldots, |P|\}$ *indicating the position of the element* a *in each linear extension (*1 *being associated with the top and* $|P|$ *with the bottom of the chain). We define* $\mathcal{A}_{\leq} = \{X_{x_1}, \ldots, X_{x_{|P|}}\}$ *to be the set of all such random variables. Obviously, the random variables in* $\mathcal{A}_{\leq}$ *are not independent.*

In this section, we prove that there exists a correspondence between the three methods for rankings elements of a poset described in Section 3 and the three stochastic orderings described in Section 4.

5.1 Averaged rankings and expected value

In the following, we show the similarities between the method of averaged rankings for ranking the elements of a poset and the stochastic ordering of expected value. For this aim we just need to realize that the position $\text{Pos}_e(y)$ *of an element* y *in the linear extension* e *coincides with* $X_y(e)$.

Theorem 1. *Let* $(P, \leq)$ *be a poset. There exists a correspondence between the weak order* $\precsim_{\text{av}}$ *on* P *provided by the method of averaged rankings and the weak order* $\succsim_{\text{EV}}$ *on* $\mathcal{A}_{\leq}$ *provided by expected value.*

Proof. It suffices to see that, for any $y \in P$, *it holds that*

$$\text{av}(y) = \frac{1}{|E_{\leq}|} \sum_{e \in E_{\leq}} \text{Pos}_e(y) = \frac{1}{|E_{\leq}|} \sum_{e \in E_{\leq}} X_y(e) = E(X_y)\,.$$

Therefore, it holds that the position of y *in* $\precsim_{\text{av}}$ *coincides with that of* X_y *in* $\succsim_{\text{EV}}$. ■

5.2 Mutual rank probabilities and statistical preference

In the following, we show the similarities between the method of mutual rank probabilities for ranking the elements of a poset and the stochastic ordering of statistical preference. For this aim we just need to realize that the mutual rank probability $p_{y<x}$ *of* x *over* y *coincides with* $Q(X_x, X_y)$.

Theorem 2. *Let $(P, \leq)$ be a poset. There exists a correspondence between the binary relation $\precsim_{\mathrm{mr}}$ on P provided by the method of mutual rank probabilities and the binary relation $\succsim_{\mathrm{SP}}$ on $\mathcal{A}_{\leq}$ provided by statistical preference.*

Proof. It suffices to see that, for any distinct $x, y \in P$, it holds that

$$\begin{aligned} p_{y<x} &= \frac{|\{e \in E_{\leq} \mid y \leq_e x\}|}{|E_{\leq}|} = \frac{\sum_{e \in E_{\leq}} \mathbb{1}(X_x(e) > X_y(e))}{|E_{\leq}|} \\ &= \pi(X_y < X_x) = Q(X_x, X_y)\,, \end{aligned}$$

since X_x and X_y are distinct if x and y are distinct. Finally, it holds that $p_{y<x} \leq \frac{1}{2}$ if and only if $Q(X_x, X_y) \leq \frac{1}{2}$, and, thus, $x \precsim_{\mathrm{mr}} y$ if and only if $X_y \succsim_{\mathrm{SP}} X_x$. ■

5.3 Maximal method and probabilistic preference

Finally, we show the similarities between the maximal method for ranking the elements of a poset and the stochastic ordering provided by probabilistic preference. For this aim we just need to realize that the sets M_i and $\mathcal{A}_i$ coincide for any i.

Theorem 3. *Let $(P, \leq)$ be a poset. There exists a correspondence between a linear extension of the weak order $\precsim_{\max}$ on P provided by the maximal method and the weak order $\succsim_{\mathrm{PP}}$ on $\mathcal{A}_{\leq}$ provided by probabilistic preference.*

Proof. It suffices to see that

$$\begin{aligned} M_1 &= \{a \in P \mid (\nexists b \in P)(a \leq b)\} \\ &= \{a \in P \mid (\exists e \in E_{\leq})(\mathrm{Pos}_e(a) = 1)\} \\ &= \{a \in P \mid \pi\big((\forall W \in \mathcal{A} \backslash \{X_a\})(X_a > W)\big) > 0\} \\ &= \{a \in P \mid \Pi_{\mathcal{A}}(X_a) > 0\} = \mathcal{A}_1\,. \end{aligned}$$

Similarly, one could prove that $M_i = \mathcal{A}_i$, for any i. Since $a \in M_i$ if and only if $X_a \in \mathcal{A}_i$, it follows that $\succsim_{PP}$ is a complete extension of $\precsim_{\max}$. ■

6 Conclusion

In the present paper we have shown how three popular methods for ranking the elements of a poset (namely averaged rankings, mutual rank probabilities and maximal method) are very closely related to three popular stochastic orderings (namely expected value, statistical preference and probabilistic preference). This work brings up some parallelisms between the interests of fields as varied as Discrete Mathematics and Statistics.

In fact, this connection we have established allows to use any stochastic ordering as a method for ranking elements of a poset. Somewhat surprisingly, this *connection may have an important application in Chemistry, where the need of establishing a weak order in a poset naturally appears (see for example [3, 4, 7, 8]).*

The next step in this investigation should be to unify this paper with our previous contribution [15] where we established a connection between voting procedures and stochastic orderings. This would allow us to conclude that finding the winner in an election, weakly ordering the elements of a poset and ordering random variables are closely related mathematical problems. In addition, it would be worth performing experimental studies on real or simulated data for comparing the methods.

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