Lot Sizing Problem Under Lead-time Uncertainty*

Romain Guillaume[1][0000-0003-4978-630X], Adam Kasperski[2][0000-0003-1956-1812], and Paweł Zieliński[2][0000-0002-1466-2887]

[1] Université de Toulouse-IRIT, Toulouse, France romain.guillaume@irit.fr
[2] Wrocław University of Science and Technology, Wrocław, Poland
{adam.kasperski, pawel.zielinski}@pwr.edu.pl

Abstract. In the supply chain, lead time uncertainty affects the effectiveness of planning. This paper discusses a lot sizing problem with uncertain lead times modelled by intervals. First, we propose to evaluate the impact of uncertainty on a given production plan by computing a best and a worst production plan over all lead time scenarios. Then, a method based on R^*_e is proposed for choosing a compromise production plan. Some methods for solving the problems based on mixed integer programming formulations are proposed. Finally, the results are illustrated and discussed using an example.

Keywords: Production planning · Lot sizing · Lead time uncertainty · Robust optimization.

1 Introduction

In supply chain planning, effective replenishment is a crucial problem. Uncertainty on lead times can be explained by variability in the supplier's actual workload (when a supplier furnishes several customers, its workload depends on the lead time of all customer orders; if total demand exceeds production capacity, lead time increases). There are many other external factors that increase the uncertainty of lead times: outsourced production overseas may introduce some randomness due to shipping perturbations, orders may not arrive on time due to work stoppages, orders depend on weather, etc. [**?**].

There are two approaches in the literature that take into account uncertainty of lead times. The first is based on the settings of Material Requirements Planning (MRP) systems, and the second is based on the optimization of lot sizing. MRP is common in developing production plans in discrete parts manufacturing. The main parameters are a lot size, a planned lead time and a safety stock. Inappropriate parameters can lead to overstocking or out-of-stock situations. In [**?**] optimization of the planned lead time for the MRP system was proposed, when the lead time uncertainty is represented by an interval. The paper [**?**] focuses

* Adam Kasperski and Paweł Zieliński were supported by the National Science Centre, Poland, grant 2022/45/B/HS4/00355

on planned lead time and lot size and studies the impact of random lead time in a single stage production system. The papers [?] and [?] concentrate on the propagation and evaluation of uncertainty, when the uncertainty on lead time is represented by a possibility distribution. The approach based on inventory or lot-sizing problem integrates uncertainty directly into the optimization problem, and the decision relates directly to a quantity by period. In this context, the problem can be decomposed into an approach accepting crossover of order [?], [?] and the approach without crossover of order [?]. The crossover is justified when each order can be delivered by different suppliers [?], while for one supplier the crossover is not considered [?]. In [?] a robust min–max lot sizing problem with discrete lead time scenarios with crossover has been recently considered. It has been shown that robust lot sizing problem is NP-hard for discrete lead time scenarios. In [?] an iterative algorithm for robust min-max lot sizing problem was proposed, with interval lead time uncertainty without crossover and possibility of splitting the orders.

In this paper we consider a single-item lot sizing problem with lead times. We seek a production plan that minimizes the total setup, production, inventory, and backordering cost. We model the lead time uncertainty by using intervals. Namely, for each period in which a production occurs, a minimum and a maximum value of lead time is provided. We discuss a model without crossover and without the possibility of splitting the orders. Several approaches exist to take uncertainty into account in optimization problems, depending on the knowledge and behavior of decision maker (such as risk or opportunity loss attitude). In our model, no additional information (such as probability or possibility distribution for uncertain lead times) is provided. In this context, the min-max criterion is often used, i.e. a solution is computed under the assumption that a worst lead time scenario will occur. This approach can model a risk-averse behavior, but can also lead to very conservative decisions. Several criteria have been proposed to soften the min-max one, such as the Hurwicz [?], Ordering Weighted Averaging [?], τ-anchor [?], or R^* [?]. In this paper we will use the R^* criterion, since it satisfies the dynamic consistency and the weak Pareto property in the bi-objective view of optimistic/pessimistic criteria, while criteria such as the Hurwicz one, do not satisfy them (see [?] for a deeper discussion).

This paper is organized as follows. In Section **??** we recall a formulation of the deterministic lot-sizing problem with lead times on production quantities. In Section **??** we propose a model of uncertainty for lead times. We introduce a scenario set containing all possible lead time scenarios. We also propose a method to choose a solution using the R^* criterion. In Section **??** we show how to compute a worst and a best lead time scenario in polynomial time, by solving a longest (shortest) path problem in a layered network. In Sections **??** and **??** we show some methods for computing a worst and a best production plan. These methods are based on mixed integer programming formulations. Finally, in Section **??**, an algorithm for solving the problem with the R^* criterion is proposed. This algorithm is illustrated with an example.

2 Deterministic Lot-Sizing Problem with Lead Times

We are given a set $[T] = \{1, \ldots, T\}$ of *periods* and a set $[T^+] = \{T+1, \ldots, T^+\}$ of *future periods*. We will also use the notation $[T-1] = \{1, \ldots, T-1\}$ and $[T^+ - 1] = \{T+1, \ldots, T^+ - 1\}$. For each future period $t \in [T^+]$, a *demand* $d_t \geq 0$ occurs. We are also given *setup* and *production* costs c_t^S, c_t^P for each period $t \in [T]$, and *inventory* and *backordering* costs c^I, c^B, which are assumed to be fixed in all future periods $t \in [T^+]$. Let $\boldsymbol{x} = (x_t)_{t \in [T]}$ be a *production plan*, where $x_t \geq 0$ is the amount of production in period $t \in [T]$. Let us denote by $\mathbb{X}$ the set of *feasible production plans*. We assume that $\mathbb{X}$ is described by some linear constraints on $\boldsymbol{x}$. For example, the minimal capacity b_t^l and the maximal capacity b_t^u on production in period $t \in [T]$ can be imposed, which yields

$$\mathbb{X} = \{\boldsymbol{x} \in \mathbb{R}_+^T : \ b_t^l \leq x_t \leq b_t^u, \ t \in [T]\}.$$

The production plan $\boldsymbol{x}$ generates a *procurement (delivery) plan* in the future periods $\boldsymbol{o} = (o_t)_{t \in [T^+]}$ by applying *lead times* $LT(t)$ to $\boldsymbol{x}$, i.e. $o_{t+LT(t)} = x_t$, $t \in [T]$, $t + LT(t) \in [T^+]$. The lead times $L(t)$, $t \in [T]$, are positive integers. We assume, like in [**?**] and [**?**], that production in period $t \in [T]$ cannot arrive after the production in the next period $(t+1) \in [T]$, which can be expressed as

$$t + LT(t) \leq (t+1) + LT(t+1), \ t \in [T-1]. \tag{1}$$

If $o_t > d_t$, then an inventory cost occurs; if $o_t < d_t$, then a backordering cost occurs in $t \in [T^+]$. The problem is to find a feasible production plan that minimizes the total setup, inventory, and backordering cost. Let us denote by $\delta_{i,t}$, $i \in [T]$, $t \in [T^+]$, a lead time parameter defined as follows:

$$\delta_{i,t} = \begin{cases} 1 \text{ if } i + LT(i) \leq t \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

If $\delta_{i,t} = 1$, then the production (order placed) in period $i \in [T]$ has arrived by period $t \in [T^+]$. If $\delta_{i,t} = 0$, then the production in period i has not yet arrived by period t. The problem can be modeled by the following mixed integer program:

$$\min \sum_{t \in [T]} (c_t^S y_t + c_t^P x_t) + \sum_{t \in [T^+]} (c^I I_t + c^B B_t) \tag{3}$$

$$\text{s.t. } B_t - I_t = D_t - O_t \qquad t \in [T^+] \tag{4}$$

$$O_t = \sum_{i \in [T]} \delta_{i,t} x_i \qquad t \in [T^+] \tag{5}$$

$$x_t \leq M y_t \qquad t \in [T] \tag{6}$$

$$B_t, I_t \geq 0 \qquad t \in [T^+] \tag{7}$$

$$y_t \in \{0,1\} \qquad t \in [T] \tag{8}$$

$$\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}_+^T, \tag{9}$$

where $D_t = \sum_{i\in[t]} d_i$ the *cumulative demand* up to period t, I_t, B_t and O_t are the *inventory*, *backordering* and the *cumulative delivery* quantity at period t, respectively. Equation (**??**) is a flow conservation constraint, equation (**??**) computes the cumulative delivery quantities in $[T^+]$ from the lead times and the production quantities in $[T]$.

3 Lot-sizing problem with uncertain lead times

Let $\widehat{LT} \geq 1$ be a common *nominal lead time*, which does not depend on periods in $[T]$. For each period $t \in [T]$, deviations $\zeta^+(t)$, $\zeta^-(t)$ from $\widehat{LT}$ are specified. The values of $\zeta^+(t)$, $\zeta^-(t)$ are nonnegative integers. Hence $LT(t) \in \{LT_{\min}(t), \dots, LT_{\max}(t)\}$, $t \in [T]$, with $LT_{\min}(t) = \widehat{LT} - \zeta^-(t)$ and $LT_{\max}(t) = \widehat{LT} + \zeta^+(t)$. We assume that $t + LT_{\min}(t) > T$ and $t + LT_{\max}(t) \leq T^+$ for each $t \in [T]$, so production at period $t \in [T]$ must arrive at some future period. We also assume that $1 + LT_{\min}(1) = T + 1$, so the production at period 1 is ready at the first future period. We will further assume that $t+LT_{\min}(t) \leq (t+1)+LT_{\min}(t+1)$ and $t + LT_{\max}(t) \leq (t + 1) + LT_{\max}(t + 1)$ for each $t \in [T - 1]$, which follows from the assumption (**??**) about the lead times imposed in Section **??**.

Let us introduce a *lead time uncertainty set* $\mathcal{L} \subseteq \{0,1\}^T \times \{0,1\}^{T^+}$, defined by a system of the following constraints:

$$\delta_{i,t} \leq \delta_{i,t+1} \qquad i \in [T], t \in [T^+ - 1] \tag{10}$$

$$\delta_{i+1,t} \leq \delta_{i,t} \qquad i \in [T - 1], t \in [T^+], \tag{11}$$

$$\delta_{t,t+LT_{\min}(t)-1} = 0 \qquad t \in [T] : t + LT_{\min}(t) - 1 > T \tag{12}$$

$$\delta_{t,t+LT_{\max}(t)} = 1 \qquad t \in [T] \tag{13}$$

$$\delta_{i,t} \in \{0,1\} \qquad i \in [T], t \in [T^+] \tag{14}$$

Constraints (**??**) mean that if a production in period $i \in [T]$ is available in $t \in [T^+]$, then it is also available in the subsequent period $t + 1 \in [T^+]$. Constraints (**??**) mean that if a production in period $i \in [T]$ is not available in $t \in [T^+]$, then production in the subsequent period $i + 1 \in [T]$ is also not available in t. Finally, constraints (**??**) and (**??**) model the minimum and the maximum lead time for each period $t \in [T]$. Notice that $\mathcal{L}$ is a discrete uncertainty set, which means that an order is delivered by the supplier if and only if it is completed. Under the assumption that all periods $t \in [T]$ have a common nominal lead time $\widehat{LT}$, we get $\mathcal{L} \neq \emptyset$. Indeed, we obtain a feasible lead time scenario by fixing $\delta_{i,t} = 1$ if $t \geq i + \widehat{LT}$ and $\delta_{i,t} = 0$, otherwise for each $i \in [T]$, $t \in [T^+]$. Having a lead time scenario $\boldsymbol{\delta} \in \mathcal{L}$, we can easily compute the lead times $L(i) = \min\{t - i : t \in [T^+], \delta_{i,t} = 1\}$ for each $i \in [T]$.

Let us denote by $\mathcal{C}(\boldsymbol{x}, \boldsymbol{\delta})$ the cost of the production plan $\boldsymbol{x} \in \mathbb{X}$ under the lead time scenario $\boldsymbol{\delta} \in \mathcal{L}$. Namely, $\mathcal{C}(\boldsymbol{x}, \boldsymbol{\delta})$ is the value of the objective function (**??**) for a fixed $\boldsymbol{x}$ and $\boldsymbol{\delta}$ (notice that $(y_t)_{t\in[T]}$, $(I_t)_{t\in[T^+]}$ and $(B_t)_{t\in[T^+]}$ are imposed by $\boldsymbol{x}$ and $\boldsymbol{\delta}$). In the following, we are interested in computing an optimistic production plan $\boldsymbol{x}^o$ and a pessimistic production plan $\boldsymbol{x}^p$, by solving the following two optimization problems:

$$\boldsymbol{x}^o = \arg\min_{\boldsymbol{x}\in\mathbb{X}} \min_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}) \tag{15}$$

$$\boldsymbol{x}^p = \arg\min_{\boldsymbol{x}\in\mathbb{X}} \max_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}) \tag{16}$$

$\boldsymbol{x}^o$ and $\boldsymbol{x}^p$ are the two extreme strategies to face to the uncertainty. In order to choose a solution taking into account a risk trade-off we can use a criterion proposed in [?]. Namely, we consider the following R^*_e problem. Given a threshold value $e \in \mathbb{R}\cup\{+\infty,-\infty\}$, if $\max_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x}^p,\boldsymbol{\delta}) > e$, then we choose the pessimistic production plan $\boldsymbol{x}^p$. Otherwise, we solve

$$\begin{aligned} \min\ & \min_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}) \\ \text{s.t.}\ & \max_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x},\boldsymbol{\delta}) \le e \\ & \boldsymbol{x}\in\mathbb{X} \end{aligned} \tag{17}$$

Hence, we apply the optimistic strategy, but with a robust constraint ensuring that the largest cost of production plan will not exceed e. The R^* criterion generalizes pessimistic and optimistic strategies. If $e=-\infty$ then we choose $\boldsymbol{x}^p$, if $e=+\infty$ then we choose $\boldsymbol{x}^o$ (model (**??**) reduces then to (**??**)). A motivation to use the R^* criterion can be found in [?]. The parameter e can be used in two ways. In our context, it is the maximum acceptable cost of a production plan, or it can be used in a sensitivity analysis. We discuss its use for sensitivity analysis in Section **??**.

4 Solving the Adversarial Problems

In this section, we consider the *adversarial problem*. Namely, we seek a best and a worst lead time scenario $\boldsymbol{\delta}\in\mathcal{L}$ for a given production plan $\boldsymbol{x}\in\mathbb{X}$, i.e., scenarios which result in the largest and the smallest total cost of $\boldsymbol{x}$. We start with a characterization of all possible cumulative deliveries in a given period $t\in[T^+]$. Let us define $I(t)=\max\{i\in[T] | i+LT_{\max}(i)\le t\}$ and $J(t)=\max\{i\in [T] | i+LT_{\min}(i)\le t\}$. Observe that the production in periods $1,\dots,I(t)$ must arrive by t and the production in periods $1,\dots,J(t)$ may arrive by t. We fix $I(t)=0$ if $i+LT_{\max}(i)>t$ for each $i\in[T]$. Observe that $I(t)\le J(t)$ and $J(t)\ge 1$ for each $t\in[T^+]$, by the assumptions made in Section **??**.

Proposition 1. *For each period* $t\in[T^+]$

$$O_t \in \left\{ \sum_{i=1}^{I(t)} x_i,\ \sum_{i=1}^{I(t)+1} x_i, \dots, \sum_{i=1}^{J(t)} x_i \right\} = \mathcal{O}_t, \tag{18}$$

where $\sum_{i=1}^{I(t)} x_i = 0$ *if* $I(t)=0$.

Proof. Let $\boldsymbol{\delta} \in \mathcal{L}$ be a lead time scenario, which defines feasible lead times $LT(i)$, $i \in [T]$. By (??), the cumulative delivery in period $t \in [T^+]$ equals

$$O_t = \sum_{\{i \in [T] : i + LT(i) \leq t\}} x_i.$$

Let $\underline{i} = \min\{i \in [T] : i + LT(i) \leq t\}$ and $\bar{i} = \max\{i \in [T] : i + LT(i) \leq t\}$. We set $\underline{i} = \bar{i} = 0$ if $i + LT(i) > t$ for each $t \in [T]$. It is easy to see that $I(t) \leq \underline{i} \leq \bar{i} \leq J(t)$, where $I(t) \leq J(t)$. By the definition of $\mathcal{L}$, $\underline{i} \in \{0, 1\}$. Indeed, if $\underline{i} > 1$, then production from some period $i > 1$ arrived before production from period 1. If $\underline{i} = 0$, then $I(t) = 0$ and $O_t = \sum_{i=1}^{0} x_i = 0 \in \mathcal{O}_t$. If $\underline{i} = 1$, then $O_t = \sum_{i=1}^{\bar{i}} x_i$, where $\bar{i} \in \{I(t), I(t)+1, \ldots, J(t)\}$. Therefore, $O_t \in \mathcal{O}_t$ and the proposition follows.

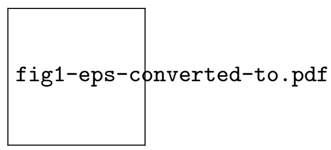

Fig. 1. A sample network for $T = 3$, $T^+ = 9$, $\boldsymbol{x} = (x_1, x_2, x_3)$, $\widehat{LT} = 4$, $LT(1) \in \{3, 4, 5\}$, $LT(2) = \{4, 5\}$, $LT(3) = \{3, 4, 5, 6\}$. The bold path corresponds to lead times $LT(1) = 4$, $LT(2) = 5$, $LT(3) = 4$.

Using Proposition ?? we can construct a layered digraph $G(V, A)$ that represents all possible lead time scenarios. The set of nodes V is partitioned into disjoint layers $V_{T+1}, \ldots, V_{T^+}$ that correspond to the future periods $t \in [T^+]$. The nodes of the layer V_t correspond to all possible cumulative delivers in $\mathcal{O}_t$, described in Proposition ??. Namely, node v_U^t in layer V_t corresponds to period $t \in [T^+]$ and U is the set of summation indices in (??), which yields the cumulative delivery $x(U) = \sum_{i \in U} x_i$ at period t. An arc exists from v_W^t to v_U^{t+1} if $W \subseteq U$. We add a starting node $\mathfrak{s} = v_\emptyset^T$ linked to all nodes of layer V_1 and a destination node $\mathfrak{t} = v_\emptyset^{T^+ + 1}$ such that all the nodes in the last layer V_{T^+} are linked to it. The cost of arc (v_W^t, v_U^{t+1}) is equal to $\max\{c^I(x(U) - D_{t+1}), c^B(D_{t+1} - x(U))\}$. A sample construction is shown in Figure ??. For example, in period $6 \in [T^+]$, we have $I(6) = 1$ (production in period 1 must be delivered by 6) and $J(6) = 3$ (productions in periods $1, 2, 3$ can be delivered by 6). In what follows, layer V_6 contains three nodes $v_{\{1\}}^6$, $v_{\{1,2\}}^6$, $v_{\{1,2,3\}}^6$ that represent three possible cumulative deliveries $O_6 = x_1$, $O_6 = x_1 + x_2$, $O_6 = x_1 + x_2 + x_3$.

Each $\mathfrak{s} - \mathfrak{t}$ path in G models a possible cumulative delivery plan $\boldsymbol{O}$. For example, the bold path shown in Figure ?? represents the cumulative delivery plan $\boldsymbol{O}$ such that $O_4 = 0$, $O_5 = x_1$, $O_6 = x_1$, $O_7 = x_1 + x_2 + x_3$, $O_8 = x_1 + x_2 + x_3$, $O_9 = x_1 + x_2 + x_3$. This plan corresponds to feasible lead times $L(1) = 4$, $L(2) = 5$, and $L(3) = 4$. The length of this path is the total inventory and backordering cost of $\boldsymbol{O}$. Because the production and setup cost of $\boldsymbol{x}$ are fixed, the shortest $\mathfrak{s}-\mathfrak{t}$ path models the best lead time scenario, while the longest $\mathfrak{s}-\mathfrak{t}$ path models the worst lead time scenario for $\boldsymbol{x}$. The graph G has $O(T\cdot(T^+-T))$ nodes. Hence, a best and a worst lead time scenarios can be computed in polynomial time by using any algorithm for the shortest (longest) path problem in acyclic digraphs (see, e.g. [?]).

5 Computing a Pessimistic Production Plan

In this section, we show a compact mixed integer programming formulation for solving the problem (??), i.e for computing a pessimistic production plan $\boldsymbol{x}^p$. The idea is to apply a dual reformulation for the longest path problem in the network $G=(V,A)$ constructed in the previous section. Let us define the variable $v_U^t \geq 0$ for each node of G. The MIP formulation is as follows:

$$
\begin{aligned}
\min\ & v_\emptyset^{T^++1} + \sum_{t\in T}(c_t^S y_t + c_t^P x_t) && && (19)\\
\text{s.t.}\ & v_\emptyset^T = 0 && && (20)\\
& v_U^{t+1} - v_W^t \geq c^I(x(U) - D_{t+1}) && \forall (v_W^t, v_U^{t+1}) \in A && (21)\\
& v_U^{t+1} - v_W^t \geq c^B(D_{t+1} - x(U)) && \forall (v_W^t, v_U^{t+1}) \in A && (22)\\
& x_t \leq M y_t && \forall t \in [T] && (23)\\
& y_t \in \{0,1\} && \forall t \in [T] && (24)\\
& v_U^t \geq 0 && \forall v_U^t \in V && (25)\\
& \boldsymbol{x} \in \mathbb{X} && && (26)
\end{aligned}
$$

For a fixed $\boldsymbol{x} \in \mathbb{X}$, the value of $v_\emptyset^{T^++1}$ is the length of the longest path in G. This length represents the total inventory and backordering cost under the worst lead time scenario. Hence, the objective function of the model expresses the total cost of $\boldsymbol{x}$ under the worst lead time scenario in $\mathcal{L}$. If all the setup costs $c_t^S = 0$, $t \in [T]$, then we get a linear programming problem. In this case, the problem can be solved in polynomial time.

Proposition 2. *If $c_t^S = 0$ for each $t \in [T]$, the a pessimistic production plan $\boldsymbol{x}^p$ can be found in polynomial time.*

6 Computing an Optimistic Production Plan

The computation of the optimistic production plan $\boldsymbol{x}^o$, i.e. solving the problem (??) can be more complex. We will propose a compact MIP model for this problem, which is based on a minimum cost flow formulation with binary variables on some arcs. Let us build a network $G'=(V',A')$ as follows. The set of nodes $V'=\{u_1,\ldots,u_T,v_{T+1},\ldots,v_{T^+},\mathfrak{s}\}$. Node $\mathfrak{s}$ has supply $\sum_{i\in[T^+]} d_i$, node

v_i has demand d_i, $i \in [T^+]$, and nodes u_i, $i \in [T]$, have supply/demand equal to 0. The set of arcs A includes arcs $(\mathfrak{s}, u_i)$, $i \in [T]$, with costs c_i^P, arcs (u_i, v_j) if $j \in \mathcal{P}_i = \{i+LT_{\min}(i), \ldots, i+LT_{\max}(i)\}$ with costs 0, arcs (v_i, v_{i+1}), $i \in [T^+-1]$ with costs c^I and arcs (v_{i+1}, v_i), $i \in [T^+-1]$ with costs c^B. A sample construction is shown in Figure **??**.

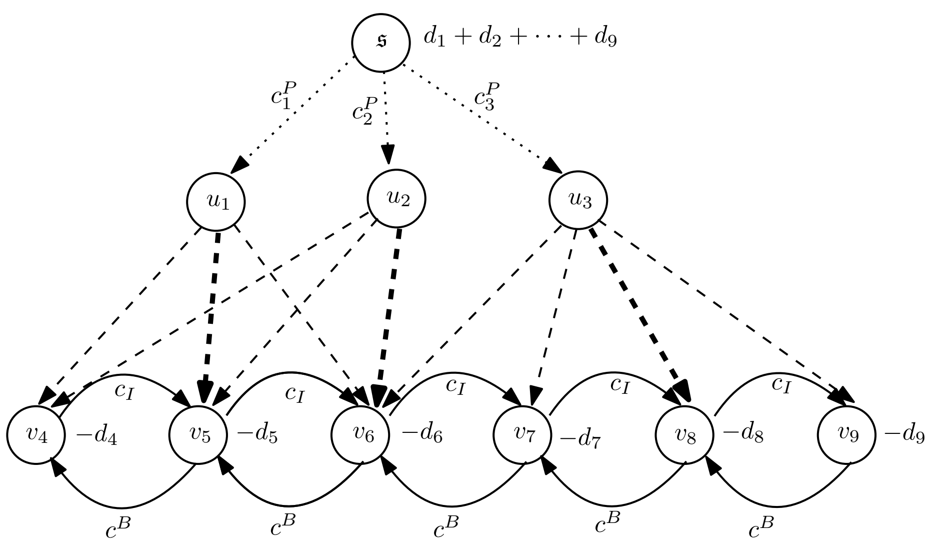

Fig. 2. A sample network for $T = 3$, $T^+ = 9$,, $\widehat{LT} = 4$, $LT(1) \in \{3, 4, 5\}$, $LT(2) = \{2, 3, 4\}$, $LT(3) = \{3, 4, 5, 6\}$. The bold dashed arcs correspond to lead times $LT(1) = 4$, $LT(2) = 4$, $LT(3) = 5$.

Let $f(u, v) \geq 0$ be a flow on arc $(u, v) \in A'$ and $c(u, v)$ be the cost of $(u, v) \in A'$. The problem can be solved by using the following mixed integer programming formulation:

$$\min \sum_{(u,v)\in A'} c(u,v)\cdot f(u,v) + \sum_{t\in[T]} c_t^S y_i \tag{27}$$

$$\text{s.t.} \sum_{j\in\mathcal{P}_i} \rho_{i,j} = 1 \qquad \forall i \in [T] \tag{28}$$

$$\rho_{i,j} + \rho_{i+1,k} \le 1 \qquad \forall j \in \mathcal{P}_i, k \in \mathcal{P}_{i+1} : k < j \tag{29}$$

$$f(\mathfrak{s}, u_i) = x_t \qquad \forall t \in [T] \tag{30}$$

$$f(u_i, v_j) \le \delta_{i,j} M \qquad \forall i \in [T], j \in \mathcal{P}_i \tag{31}$$

$$x_t \le M y_t \qquad t \in [T] \tag{32}$$

$$\text{Flow mass balance constraints} \tag{33}$$

$$y_t \in \{0,1\} \qquad t \in [T] \tag{34}$$

$$\rho_{i,j} \in \{0,1\} \qquad i \in [T], j \in \mathcal{P}_i \tag{35}$$

$$\boldsymbol{x} \in \mathbb{X} \tag{36}$$

Binary variables $\rho_{i,j}$ model feasible lead times. Namely, $\rho_{i,j} = 1$ if production in period $i \in [T]$ arrives in period $j \in \mathcal{P}_i$. Constraints (**??**) ensure that the production in $i \in [T]$ arrives in exactly one of the possible periods in $\mathcal{P}_i \subseteq [T^+]$. Constraints (**??**) ensure that production in period i cannot arrive after production in period $i+1$. In the example in Figure **??**, we have $\mathcal{P}_1 = \{4,5,6\}$, $\mathcal{P}_2 = \{4,5,6\}$. Therefore, to avoid incorrect lead times we add constraints $\delta_{1,5} + \delta_{2,4} \le 1$, $\delta_{1,6} + \delta_{2,4} \le 1$ and $\delta_{1,6} + \delta_{2,5} \le 1$. Constraints (**??**) fix the flow on arcs $(\mathfrak{s}, u_i)$ equal to the production x_i, $i \in [T]$. Constraints (**??**) ensure that production x_i arrives at the correct period $j \in [T^+]$. Constraints (**??**) are standard mass balance flow constraints for network G (see, e.g. [**?**]).

For each feasible choice of $\rho_{i,j}$, the corresponding flow $f(u,v)$ models a delivery plan with the total production, inventory and backordering cost equal to the cost of the flow. Adding the setup costs, we get the total cost of production plan $\boldsymbol{x}$. Hence, an optimal solution to (**??**)-(**??**) represents the cheapest (optimistic) production plan under the beast lead time scenario. Observe that computing $\boldsymbol{x}^o$ requires binary variables even if all setup costs are equal to 0.

7 Computing an R_e^* Production Plan

In this section, we propose a method of solving the R_e^* problem. We will use the compact MIP formulations constructed in Section **??** and Section **??**. We start by solving the formulation (**??**)-(**??**). We do not have to solve it to optimality. It is enough to find a production plan $\boldsymbol{x}'$ feasible to (**??**)-(**??**), satisfying the constraint

$$v_\emptyset^{T^+ + 1} + \sum_{t\in T} (c_t^S y_t + c_t^P x_t') \le e$$

If no such a solution exists, then w choose the pessimistic plan $\boldsymbol{x}^p$ by solving (**??**)-(**??**) to optimality. Otherwise, we solve the MIP formulation (**??**)-(**??**) that is equivalent to (??). To speed up computations, we can fix $\boldsymbol{x'}$ as a starting solution in the solver.

$$\min \sum_{(u,v)\in A'} c(u,v)\cdot f(u,v) + \sum_{t\in[T]} c_t^S y_i \tag{37}$$

$$\text{s.t. } v_\emptyset^{T^+ +1} + \sum_{t\in T}(c_t^S y_t + c_t^P x_t) \le e \tag{38}$$

$$\text{Constraints (??)} - \text{(??)} \tag{39}$$

$$\text{Constraints (??)} - \text{(??)} \tag{40}$$

$$x_t \le M y_t \qquad t\in[T] \tag{41}$$

$$y_t \in \{0,1\} \qquad t\in[T] \tag{42}$$

$$\rho_{i,j} \in \{0,1\} \qquad i\in[T], j\in\mathcal{P}_i \tag{43}$$

$$\boldsymbol{x} \in \mathbb{X} \tag{44}$$

Notice that using a similar idea, we can build a model with the Hurwicz criterion (see, e.g [?]). The objective function is then a convex combination of (??) and the left-hand side of (??) with constraints (??)-(??).

Example 1 *Consider a planning problem with horizon* $T=5$, $T^+=7$ *and* $\mathbb{X} = \prod_{t\in[T]}[0,b_t^u]$, *where* $\boldsymbol{b}^u = (30,20,15,30,30)$, $\boldsymbol{d} = (10,15,30,15,20)$, $\boldsymbol{LT}_{min} = (1,1,1,1,1,1)$, $\boldsymbol{LT}_{max} = (1,2,3,2,2,2)$, *and the costs* $c^I = 1, c^B = 2$ $c_t^S = 10, c_t^P = 1.5, \forall t\in[T]$. *Figure* **??***(a) shows the best cost of production plan,*

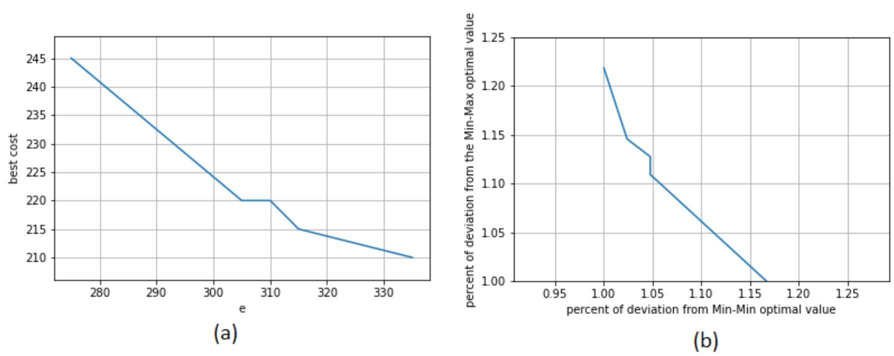

Fig. 3. Compromised solutions

depending on parameter e, *which corresponds to the Pareto front (PF) with pessimistic/optimistic criteria. As expected, we can see that if we increase the robustness of a plan, we decrease the possibility of having a low cost. We can note that this PF is not convex, so using the Hurwicz criterion, which is a convex* *combination of the pessimistic/optimistic criteria, cannot lead to a compromised solution obtained by using R_e^* approach. Figure* **??***(b) shows the set of solutions depending on* e*. The x-axis is the deviation from the optimal cost with optimistic criteria* $(\frac{\min_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x}^e,\boldsymbol{\delta})}{\min_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x}^o,\boldsymbol{\delta})})$ *and y-axis is the deviation from the optimal cost with pessimistic strategy* $(\frac{\max_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x}^e,\boldsymbol{\delta})}{\max_{\boldsymbol{\delta}\in\mathcal{L}} \mathcal{C}(\boldsymbol{x}^p,\boldsymbol{\delta})})$*.*

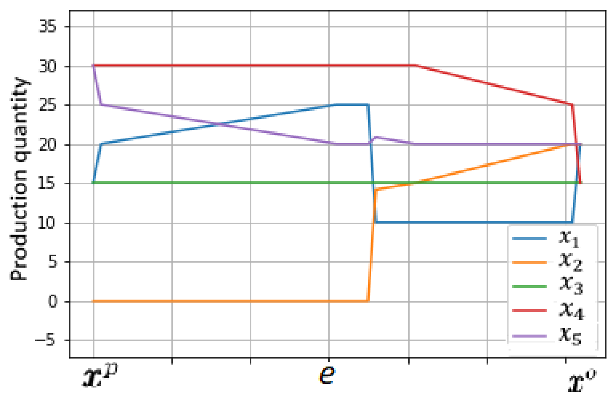

Fig. 4. Optimal solutions evolution

Figure **??** *shows the production quantity* $\boldsymbol{x}$ *depending on* e*. On the left we have a pessimistic solution* $\boldsymbol{x}^p$ *and on the right an optimistic solution* $\boldsymbol{x}^o$*. We can see that production in some periods can be zero in more pessimistic solutions while production in all periods is nonzero for more optimistic solutions. We note that optimism is taken into account either by a progressive transfer between periods, for example at the beginning between periods 5 and 1, or by breaks due to the setup cost, for example when we start producing in period 2.*

8 Conclusions

In this paper, we have studied the problem of production planning in a context of uncertain lead times. To find a production plan we have applied the R^* criterion and we have proposed an algorithm to solve the problem. We have shown that the adversarial problems (i.e. computing a best and a worst lead time scenarios for a given production plan) can be solved in polynomial time and a particular case of solving the robust pessimistic problem is also polynomially solvable. In future research, we would like to study the complexity of computing an optimistic production plan. Moreover, we have assumed that production orders cannot be split. One perspective is to study the problem in the case where splitting the orders is allowed.