Counting homomorphisms between finite Gödel algebras

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Abstract. The algebraic semantics of propositional Gödel logic is given by the variety of Gödel algebras, that is, prelinear Heyting algebras, or, equivalently, idempotent BL (or MTL) algebras. In this work we provide a recurrence that allows to compute the number of homomorphisms between finite Gödel algebras. We then show that the same recurrence can be used to compute the cardinality of the hom-sets also for the finite members of other varieties of algebras related to many-valued logics.[1]

Keywords: Gödel algebras, finite forests, homomorphisms, monoid of endomorphisms

1 Introduction

Gödel logic (or Gödel-Dummett logic) is one of the most relevant systems of intermediate, many-valued, mathematical fuzzy logic [12,11]. The Lindenbaum-Tarski algebras of propositional Gödel logic form the variety $\mathbb{G}$ of Gödel algebras.

In this paper we shall solve the problem of computing the number of homomorphisms between any two given finite Gödel algebras. To achieve this we actually deal with the analogous problem in a category dually equivalent to finite Gödel algebras, namely, *finite forests*, and we provide a recurrence relation that allows to determine the desired number recursively on simpler forests.

We shall apply our results to derive the cardinalities of finite algebras, finitely generated free algebras, and of the monoids of endomorphisms of finite algebras. These results are of direct relevance for logic as free algebras are the Lindenbaum algebras of pure propositional logic, while each finite Gödel algebra, being homomorphic image of some finite free algebra (the variety of Gödel algebras is locally finite), is the Lindenbaum algebra of a uniquely determined theory. Therefore endomorphisms of finite algebras are classes of logically equivalent (wrt. a theory) substitutions.

We generalise our result to several other varieties of algebras constituting the algebraic semantics of many-valued or non-classical logics.

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2 Preliminaries

We refer to [12] and [2] for any background on Gödel logic and Gödel algebras.

Gödel propositional logic arises by extending intuitionistic propositional logic with the axiom scheme

$$(\varphi \to \psi) \vee (\psi \to \varphi)\,.$$

Equivalently, it is obtained by extending Hájek's Basic Fuzzy Logic BL, or Esteva and Godo's Monoidal t-norm based logic MTL, with the axiom scheme

$$\varphi \to (\varphi \,\&\, \varphi)\,.$$

Gödel propositional logic is algebraisable in the sense of Lindenbaum-Tarski and Blok-Pigozzi. Its equivalent algebraic semantics is given by the variety of Gödel algebras $\mathbb{G}$.

As is well known, Gödel algebras can be equivalently thought as Heyting algebras satisfying prelinearity, that is, $(x \to y) \vee (y \to x) = 1$ or as BL-algebras (the semantics of Hájek's Basic Fuzzy Logic) satisfying idempotence, that is, $x \,\&\, x = x$. Notice that Gödel algebras arise also as idempotent MTL-algebras (the semantics of Esteva and Godo's Monoidal t-norm-based Logic [11]).

The usual signature for a Gödel algebra $\mathcal{A}$ is $\mathcal{A} = (A, \wedge, \to, 0)$, as the other operations can be derived from $\wedge, \to$ and the constant 0. As a matter of fact, the usually derived operations are $\neg x := x \to 0$, $1 := \neg 0$, $x \vee y := ((x \to y) \to y)) \wedge ((y \to x) \to x)$, $x \leftrightarrow y := (x \to y) \wedge (y \to x)$. When Gödel algebras are considered as BL-, or MTL-algebras, idempotence implies that $x \,\&\, y = x \wedge y$.

We can consider $\mathbb{G}$ as a category, where the objects are the algebras and the arrows between them are the homomorphisms. Given two Gödel algebras $\mathcal{A}$ and $\mathcal{B}$, we write as usual $Hom(\mathcal{A}, \mathcal{B})$ for the set of all homomorphisms $h \colon \mathcal{A} \to \mathcal{B}$.

The algebras in $\mathbb{G}$ of finite cardinality constitute the full subcategory $\mathbb{G}_{fin}$ of finite Gödel algebras. Clearly, if $\mathcal{A}, \mathcal{B} \in \mathbb{G}_{fin}$, then $Hom(\mathcal{A}, \mathcal{B})$ is a finite set of maps.

A *filter* R of a Gödel algebra $\mathcal{A}$ is an upward closed subset of A, that is $\uparrow R = R$, (where $\uparrow R = \{y \in A \mid \text{ there is } x \in R \text{ such that } x \leq y\}$), further closed under meets, that is, if $x, y \in R$, then $x \wedge y \in R$, too. R is *proper* iff $R \subsetneq A$. A proper filter $\mathfrak{p}$ of $\mathcal{A}$ is *prime* iff for each pair of elements $x, y \in A$, either $x \to y \in \mathfrak{p}$ or $y \to x \in \mathfrak{p}$.

The poset of prime filters of $\mathcal{A}$ ordered by *reverse* inclusion: $\mathfrak{p} \leq \mathfrak{q}$ iff $\mathfrak{q} \subseteq \mathfrak{p}$ is called the *prime spectrum* $\operatorname{Spec} \mathcal{A}$ of $\mathcal{A}$.

An element $0 \neq x \in \mathcal{A}$ is *join-irreducible* iff $x = y \vee z$ implies that $x = y$ or $x = z$. We write $JI(\mathcal{A})$ for the poset of all join-irreducible elements of $\mathcal{A}$, ordered by restriction.

A filter $R \subseteq A$ is *principal* if there is $x \in A$ such that $R = \uparrow\{x\}$. Such an element x is called the *generator* of F. All filters of any finite Gödel algebra are principal. In particular each prime filter $\mathfrak{p} \in \operatorname{Spec} \mathcal{A}$ is generated by a join-irreducible element of $\mathcal{A}$, and, *viceversa*, every join-irreducible element of $\mathcal{A}$ generates a prime filter in $\operatorname{Spec} \mathcal{A}$. That is to say, the posets $\operatorname{Spec} \mathcal{A}$ and $JI(\mathcal{A})$ are order-isomorphic for every $\mathcal{A} \in \mathbb{G}_{fin}$.

Notice that Boolean algebras are exactly the Gödel algebras satisfying *tertium non datur* ($x \vee \neg x = 1$), or, equivalently, the *double negation* law ($\neg\neg x = x$). In particular, for any finite Boolean algebra $\mathcal{B}$, we have $JI(\mathcal{B}) = At(\mathcal{B})$, the set of atoms of $\mathcal{B}$.

3 General considerations

In order to effectively compute the number of homomorphisms between two finite Gödel algebras $|Hom(\mathcal{A}, \mathcal{B})|$, we make the reasonable assumption of just having as input, or being able to compute from the input, the set of join-irreducible elements $JI(\mathcal{A})$ and $JI(\mathcal{B})$. In some cases we only need to know some characterising properties of the considered algebras to apply our methods. This is the case for instance of the computation of the cardinality of free algebras and of their hom-sets, where we only need to know the number of generators of the involved algebras. (See [4] for an analogous discussion on computing coproducts of Gödel algebras).

Determining the number of homomorphisms is a traditional problem for several classes of algebras. For the case of finite Boolean algebras, we can derive easily that $Hom(\mathcal{A}, \mathcal{B})$ has exactly $|At(\mathcal{A})|^{|At(\mathcal{B})|}$ elements, where $At(.)$ is the set of atoms of the Boolean algebra. One way to prove this fact uses the finite specialisation of Stone's duality, recalling that the category of finite Boolean algebras and homomorphisms is dually equivalent to the category of finite sets and maps between them (see [16,15] for background on categories and dual equivalences).

Theorem 1. *For each pair of finite Boolean algebras $\mathcal{A}, \mathcal{B}$, the set $Hom(\mathcal{A}, \mathcal{B})$ has exactly $|At(\mathcal{A})|^{|At(\mathcal{B})|}$ elements.*

Proof. The equivalence between finite Boolean algebras and finite sets is implemented by the contravariant functor $At\colon \mathbb{B}_{fin} \to \mathsf{Set}_{fin}$. Then we immediately have that $|Hom(\mathcal{A}, \mathcal{B})| = |Hom(At(\mathcal{B}), At(\mathcal{A}))|$.

Remark 1. Throughout this paper we shall mostly use only the object part of functors. For instance, in Theorem 1 we do not need to recall the reader the definition of $At(h)\colon At(\mathcal{B}) \to At(\mathcal{A})$ as $(At(h))(x) = y$ where y is the uniquely determined atom in $\mathcal{A}$ such that $\uparrow\{y\} = h^{-1}[\uparrow\{x\}]$. The only functorial concept about maps we shall use is that the functors realising the dual equivalences we are going to mention, are, by definition, *contravariant.*

We shall use the same approach, namely, computing the cardinality of hom-sets in a dually equivalent category, to solve the problem for the case of finite Gödel algebras, and for related structures.

From the point of view of logic, we get a series of interesting applications as corollaries.

Recall that two formulas φ, ψ are equivalent in Gödel propositional logic (written $\varphi \equiv \psi$) iff $\varphi \leftrightarrow \psi$ is a tautology. For every integer $n \geq 0$, the set of classes of logically equivalent formulas written on the first n propositional letters

$x_1, \ldots, x_n$, equipped with naturally defined operations form the *Lindembaum algebra* of the logic (on the first n variables), that in turn is isomorphic with the *free Gödel algebra* over n free generators $\mathcal{F}_n(\mathbb{G})$ (see [2]).

We can use our results to compute the cardinality of $\mathcal{F}_n(\mathbb{G})$ and of any other finite Gödel algebra $\mathcal{A}$ as, by definition of free algebra, in every variety $\mathbb{V}$ it holds that $|\mathcal{A}| = |Hom(\mathcal{F}_1(\mathbb{V}), \mathcal{A})|$ (see [4],[2] for a recurrence to compute the cardinalities of free Gödel algebras).

Further, recall that for every Gödel algebra $\mathcal{A}$, the set $Hom(\mathcal{A}, \mathcal{A})$ can be structured as a monoid $\mathcal{E}nd(\mathcal{A}) = (Hom(\mathcal{A}, \mathcal{A}), \circ, id)$, where $\circ$ is functional composition and id is the identity map. $\mathcal{E}nd(\mathcal{A})$ is the *monoid of endomorphisms* of $\mathcal{A}$. Each endomorphism of the free n-generated Gödel algebra $\mathcal{F}_n(\mathbb{G})$ can be considered a substitution, up to logical equivalence, of individual variables with terms, that is, if $\sigma \in \mathcal{E}nd(\mathcal{F}_n(\mathbb{G}))$ then, by definition of homomorphism, the action of σ is completely determined by the following data:

$$\sigma(x_1) = \varphi_1, \sigma(x_2) = \varphi_2, \ldots, \sigma(x_n) = \varphi_n \,,$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n$ are terms (formulas) over the first n individual variables $x_1, \ldots, x_n$. Notice that if φ and ψ are two formulas over $x_1, \ldots, x_n$ and $\varphi \equiv \psi$, then $\sigma(\varphi) \equiv \sigma(\psi)$, too.

Every Gödel algebra $\mathcal{A}$, being homomorphic image of a free Gödel algebra, is the Lindenbaum algebra of a uniquely determined theory[2] $\Theta(\mathcal{A})$ of propositional Gödel logic. Then, endomorphisms of $\mathcal{A}$ can be identified with logically equivalent (w.r.t. $\Theta(\mathcal{A})$) classes of substitutions, in the following sense:

$$\sigma \equiv_{\Theta(\mathcal{A})} \tau \qquad \text{if and only if} \qquad \Theta(\mathcal{A}) \models \sigma(\varphi) \leftrightarrow \tau(\varphi) \,,$$

for any formula $\varphi(x_1, \ldots, x_n)$.

The automorphisms of a Gödel algebra $\mathcal{A}$ form a group $\mathcal{A}ut(\mathcal{A})$ which is a submonoid of $\mathcal{E}nd(\mathcal{A})$. Logically speaking, an automorphism in $\mathcal{A}ut(\mathcal{A})$ is seen as an invertible substitution (or its equivalence class), that is $\sigma \in \mathcal{A}ut(\mathcal{A})$ iff there is $\sigma^{-1} \in \mathcal{E}nd(\mathcal{A})$ such that $\sigma \circ \sigma^{-1} = id = \sigma^{-1} \circ \sigma$.

The group of automorphisms of finite Gödel algebras is studied in [8], [7], while the groups of automorphisms of other classes of algebras of mathematical fuzzy logics are investigated in [1], [6].

4 Finite forests

For background on this section see [2], [4].

A *forest* $F = (F, \leq)$ is a poset such that the downset $\downarrow\{x\} = \{y \in F \mid y \leq x\}$ of each element is totally ordered. A *tree* T is a forest with a smallest element, called the *root* of T. A downward closed subposet of a forest F is called a *subforest* of F.

Given two forests F and G a map $f \colon F \to G$ is *order preserving* iff $x \leq y$ implies $f(x) \leq f(y)$. An order preserving map $g \colon F \to G$ is *open* iff for all

[2] Here we assume theories are deductively closed sets of formulas.

$y \leq f(x)$ there is $z \in F$, with $z \leq x$, such that $f(z) = y$. Equivalently, open maps carry subforests to subforests.

We denote by FF the category whose objects are finite forests, and whose morphisms are order-preserving open maps between forests. We denote by FT the full subcategory of FF having as objects the finite trees.

Let $\mathbf{1}$ denote the forest whose underlying set contains one element. Let us further denote with $F_\bot$ the tree obtained from the forest F by adding a new minimum element, that is, an element $\bot$ such that $\bot < x$ for all $x \in F$. Notice that each tree in FT can be expressed as $F_\bot$ for the forest $F = T \setminus \{r\}$, where r is the root of T. The following results can be found in [2], [4].

Theorem 2. *1. $\mathbf{1}$ is the terminal object in both* FF *and* FT.

2. *The empty forest $\emptyset$ is the initial object in* FF, *while the initial object in* FT *is* $\mathbf{1}$.
3. *The coproduct object $F + G$ of two finite forests is the forest obtained as disjoint union of F and G.*
4. *The coproduct object $F_\bot +_\bot G_\bot$ of two finite trees is isomorphic with the tree $(F + G)_\bot$ where the latter $+$ is the coproduct of forests.*
5. *In both* FF *and* FT *the product object of two trees $F_\bot \times G_\bot$ is isomorphic with the tree $((F_\bot \times G) + (F \times G) + (F \times G_\bot))_\bot$ (see [4, Lemma 4.3]).*
6. *In* FF *products objects distribute over coproduct objects, that is $F \times (G + H) \cong (F \times G) + (F \times H)$ (see [4, Lemma 4.2]).*

Notice that Theorem 2 allows to compute any finite coproduct of forests and of trees, and recursively any finite product of forests and of trees.

The *height* of an element x of a finite forest F is the cardinality of $\downarrow\{x\}$. The *height* of a finite forest F is the maximum height of the elements of F. Given a forest F and a natural number $n > 0$ we let $F^{(n)}$ be the *pruning* of F *at level* n, that is $F^{(n)}$ is the subforest of F consisting of all the elements of F of height at most n.

For any finite Gödel algebra $\mathcal{A}$, let $\operatorname{Spec} \mathcal{A}$ be the poset of prime filters of $\mathcal{A}$, ordered by *reverse* inclusion. Observe that $\operatorname{Spec} \mathcal{A}$ is order-isomorphic with the poset of join-irreducible elements of $\mathcal{A}$, ordered by the restriction of the order of $\mathcal{A}$. $\operatorname{Spec} \mathcal{A}$ is a finite forest for each $\mathcal{A} \in \mathbb{G}_{fin}$.

Further, for any pair of finite Gödel algebras $\mathcal{A}$, $\mathcal{B}$, and any homomorphism $h \colon \mathcal{A} \to \mathcal{B}$, we let $\operatorname{Spec} h \colon \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} \mathcal{A}$ be the map defined as $(\operatorname{Spec} h)(\mathfrak{p}) = h^{-1}[\mathfrak{p}]$, for any $\mathfrak{p} \in \operatorname{Spec} \mathcal{B}$.

We let $Sub(F)$ denote the set of all subforests of F. For each finite forest F we let $\operatorname{Sub} F$ be the system $(Sub(F), \cap, \to, \emptyset)$, where, for all $A, B \in Sub(F)$, $A \to B = F \setminus \uparrow(A \setminus B)$. $\operatorname{Sub} F$ is a finite Gödel algebra for every $F \in \mathsf{FF}$.

For any pair of finite forests F, G, and any morphism $f \colon F \to G$, we let $\operatorname{Sub} f \colon \operatorname{Sub} G \to \operatorname{Sub} F$ be the map defined as $(\operatorname{Sub} f)(X) = f^{-1}[X]$, for any $X \in \operatorname{Sub} G$.

With these definitions in place we can state the following.

Theorem 3. *The categories $\mathbb{G}_{fin}$ and* FF *are dually equivalent via the contravariant functors* $\operatorname{Spec} : \mathbb{G}_{fin} \to \mathsf{FF}$ *and* $\operatorname{Sub} : \mathsf{FF} \to \mathbb{G}_{fin}$.

The dual equivalence of Thm. 3 is implicit in Horn's works on Gödel algebras [13,14], called L-algebras in those papers. See [2],[10],[9] for thorough explanations.

Theorem 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite Gödel algebras. Then*

$$|Hom(\mathcal{A},\mathcal{B})| = |Hom(\mathrm{Spec}\,\mathcal{B}, \mathrm{Spec}\,\mathcal{A})|\,.$$

Proof. Immediate, from Theorem 3.

We call a forest of the form $\sum_{i=1}^{n} \mathbf{1}$ a *poset of roots*. Notice that such a poset can be safely identified with its underlying set. Further, the morphisms between any two posets of roots can be identified with the functions between the two underlying sets.

The full subcategory of finite Boolean algebras $\mathbb{B}_{fin}$ of $\mathbb{G}_{fin}$ is dually equivalent to the full subcategory of FF whose objects are posets of roots. The functor Spec, when applied to a finite Boolean algebra $\mathcal{B}$ produces a poset of roots which is isomorphic with the set $\mathcal{A}t(\mathcal{B})$ of atoms of $\mathcal{B}$.

Recall that the category Set_{fin} dual to finite Boolean algebras, has exponentiation, entailing that there are objects that encode the hom-sets. Interestingly enough, FF does not have exponential objects, but the full subcategory $\mathsf{FF}^{(2)}$ of forests of height at most 2, which is dually equivalent with the finite algebras of three-valued Gödel logic, does have exponentiation, as shown in [5].

5 Counting morphisms between finite forests

As customary in (locally small) categories, the set of morphisms $f\colon F \to G$ between two finite forests $F, G \in \mathsf{FF}$ is called the hom-set of $F \to G$ and denoted $Hom(F,G)$. This notation has been already applied in this paper for homomorphisms of Gödel algebras, and shall be applied to any other category dealt with in the sequel.

Lemma 1. *Let $F = \sum_{i=1}^{n} T_i$ and G be finite forests. Then the cardinality of $Hom(F,G)$ is given by*

$$\prod_{i=1}^{n} |Hom(T_i, G)|\,.$$

Proof. Immediate, as F is the coproduct of $T_1,\ldots,T_n$.

Notice that, assuming $T_1,\ldots,T_n$ are finite trees, Lemma 1 reduces the problem of counting morphisms between two forests to the problem of counting morphisms between a tree and a forest.

Lemma 2. *Let $T, U_1,\ldots,U_m$ be finite trees and $G = \sum_{j=1}^{m} U_i$. Then the cardinality of $Hom(T,G)$ is given by*

$$\sum_{j=1}^{m} |Hom(T, U_j)|\,.$$

Proof. Clearly, each morphism $f\colon T \to G$ must map the root of T to the root of some tree U_j in G. Then, f maps the whole of T into U_j. On the other hand, any map $f_j\colon T \to U_j$ is identifiable with a uniquely determined map $f\colon T \to G$.

Lemma 1 and Lemma 2 reduce the problem of counting morphisms between two forests to the problem of counting morphisms between two trees.

Lemma 3. *Let* $\{T_r \mid r \in \{1,\ldots,h\}\}$ *and* $\{U_s \mid s \in \{1,\ldots,k\}\}$ *be two sets of finite trees. Let further* $T = \left(\sum_{r=1}^{h} T_r\right)_\perp$ *and* $U = \left(\sum_{s=1}^{k} U_s\right)_\perp$. *Then, the cardinality of* $Hom(T,U)$ *is given by the following recurrence:*

$$\prod_{r=1}^{h}\left(|Hom(T_r,U)| + \sum_{s=1}^{k}|Hom(T_r,U_s)|\right).$$

Proof. Let $f\colon T \to U$ be a morphism, and let p_r be the root of T_r. Notice that p_r either maps to the root s of U or to the root q_s of some U_s. Then, the set of restrictions $\{f_r = f \restriction T_r \mid f\colon T \to U\}$ of the morphisms $T \to U$ to T_r is in bijective correspondence with $Hom(T_r,U) \cup \bigcup_{s=1}^{k} Hom(T_r,U_s)$, therefore it counts exactly $N_r = |Hom(T_r,U)| + \sum_{s=1}^{k}|Hom(T_r,U_s)|$ elements. For each root $p_r \in \{p_1,\ldots,p_k\}$ its image $f(p_r)$ can be choosen independently of the other roots, whence, for any two distinct indices $r \neq s \in \{1,\ldots,h\}$ the restrictions f_r and f_s can also be defined independently one from the other. We conclude $|Hom(T,U)| = \prod_{r=1}^{h} N_r$.

It may seem that the recurrence in Lemma 3 is not sufficient to compute recursively $|Hom(T,U)|$, since in the product there appears $|Hom(T_r,U)|$ which entails the counting of the morphisms from the reduced tree T_r to the unreduced tree U. As a matter of fact, the following Lemma proves that the reduction of T to T_r induces an actual reduction of U to a simpler subtree.

Lemma 4. *Let* F *and* G *be two finite forests, and let* n *be the height of* F. *Then* $Hom(F,G) = Hom(F,G^{(n)})$.

Proof. Clearly, each morphism $f\colon F \to G$ must map any element $p \in F$ to an element $f(p) \in G$ such that the height of $f(p)$ is at most the height of p, whence, the image $f[F]$ is included in $G^{(n)}$. On the other hand, each map $g\colon F \to G^{(n)}$ is a map $g\colon F \to G$.

Concluding the observation following Lemma 3, an application of Lemma 4 allows us to replace in the recurrence of Lemma 3 the summand $|Hom(T_r,U)|$ with $|Hom(T_r,U^{(t_r)})|$, where t_r is the height of T_r.

Lemma 5. *For any finite forest* F, $|Hom(F,\mathbf{1})| = 1$ *and* $|Hom(\mathbf{1},F)| = |F^{(1)}|$.

Proof. $|Hom(F,\mathbf{1})| = 1$ since $\mathbf{1}$ is the terminal object in FF. By Lemma 4, $|Hom(\mathbf{1},F)| = |Hom(\mathbf{1},F^{(1)})|$, but $F^{(1)}$ is a poset of roots, that is $F^{(1)} = \sum_{i=1}^{|F^{(1)}|}\mathbf{1}$. By Lemma 2, $|Hom(\mathbf{1},F^{(1)})| = \sum_{i=1}^{|F^{(1)}|}|Hom(\mathbf{1},\mathbf{1})| = |F^{(1)}|$.

We are ready to introduce the recurrence relation that allows to compute the number of morphisms between any two finite forests.

Theorem 5. *Let F and G be two finite forests. Then we may display*

$$F = A + \sum_{i=1}^{n} \left(\sum_{r=1}^{h_i} T_{ir} \right)_{\perp} \qquad \textit{and} \qquad G = B + \sum_{j=1}^{m} \left(\sum_{s=1}^{k_j} U_{js} \right)_{\perp} ,$$

where A and B are two finite posets of roots, while $\{T_{ir} \mid i \in \{1, \ldots, n\}, r \in \{1, \ldots, h_i\}\}$ and $\{U_{js} \mid j \in \{1, \ldots, m\}, s \in \{1, \ldots, k_j\}\}$ are two finite sets of finite trees[3]*. Then the cardinality of $Hom(F, G)$ is given by:*

$$(|B| + m)^{|A|} \prod_{i=1}^{n} \left(|B| + \sum_{j=1}^{m} \prod_{r=1}^{h_i} \left(|Hom(T_{ir}, U_j^{(t_{ir})})| + \sum_{s=1}^{k_j} |Hom(T_{ir}, U_{js})| \right) \right) ,$$

where $U_j = \left(\sum_{s=1}^{k_j} U_{js} \right)_{\perp}$ and t_{ir} is the height of T_{ir}, for each $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, h_i\}$.

Proof. Let $T = \sum_{i=1}^{n} T_i$ for $T_i = (\sum_{r=1}^{h_i} T_{ir})_{\perp}$ and $U = \sum_{j=1}^{m} U_j$. By Lemma 1, $|Hom(F, G)| = |Hom(A, G)| \cdot |Hom(T, B + U)|$. By Lemma 2, Lemma 1 and Lemma 5, $|Hom(A, G)| = (|B| + \sum_{j=1}^{m} |Hom(\mathbf{1}, U_j)|)^{|A|} = (|B| + m)^{|A|}$. Now, $|Hom(T, B+U)| = \prod_{i=1}^{n}(|Hom(T_i, B+U)|$ by Lemma 1. We proceed observing that, by Lemma 2, $|Hom(T_i, B + U)| = |Hom(T_i, B)| + |Hom(T_i, U)|$ for each $i \in \{1, \ldots, n\}$. Finally, $|Hom(T_i, B)| = |B|$ by Lemma 5, while $|Hom(T_i, U)| = \sum_{j=1}^{m} |Hom(T_i, U_j)|$ by Lemma 2, and to conclude the proof, by Lemma 3, $|Hom(T_i, U_j)| = \prod_{r=1}^{h_i}(|Hom(T_{ir}, U_j^{(t_{ir})})| + \sum_{s=1}^{k_j} |Hom(T_{ir}, U_{js})|)$.

Clearly, we can use Theorem 5 to compute the number of homomorphisms between the finite Gödel algebras $\mathcal{A}$ and $\mathcal{B}$. Indeed, by Theorem 4, we just have to apply the recurrence to $F = \mathrm{Spec}\,\mathcal{B}$, and $G = \mathrm{Spec}\,\mathcal{A}$.

Notice that when F and G are posets of roots, that is, in the notation of Theorem 5, $F = A$ and $G = B$, we have that $|Hom(F, G)| = |B|^{|A|}$. Applying the latter identity to finite Boolean algebras $\mathcal{B}_1, \mathcal{B}_2$, we get, as expected, $|Hom(\mathcal{B}_1, \mathcal{B}_2)| = |\mathcal{A}t(\mathcal{B}_1)|^{|\mathcal{A}t(\mathcal{B}_2)|}$.

In some cases we can use more direct ways to compute the number of homomorphisms than the recurrence offered by Theorem 5.

Lemma 6. *For any finite forest F:*

1. *The cardinality of $Hom(F_{\perp}, \mathbf{1}_{\perp})$ is the number of nonempty subforests of $F_{\perp}$.*
2. *The cardinality of $Hom(F, \mathbf{1} + \mathbf{1}_{\perp})$ is the number of subforests of F.*

[3] In the display of F and G we have to single out the posets of roots A and B, since by definition there are no empty trees.

Proof. Let r be the root of $\mathbf{1}_\perp$. For any nonempty subforest $G \subseteq F_\perp$ let $f_G\colon F_\perp \to \mathbf{1}_\perp$ be the map sending each element of G to r. It is clear that $G \mapsto f_G$ realises a bijection between nonempty subforests of $F_\perp$ and $Hom(F_\perp, \mathbf{1}_\perp)$, with inverse map $f \mapsto f^{-1}(r)$. By the same token, for any subforest $G \subseteq F$ let $g_G\colon F \to \mathbf{1}+\mathbf{1}_\perp$ be the map sending all and only the elements of G to r. Each element in $F \setminus G$ is mapped to the cover of r iff it is greater than some element of G. Again, $G \mapsto g_G$ bijects the subforests of F onto $Hom(F, \mathbf{1}+\mathbf{1}_\perp)$, with inverse map $g \mapsto g^{-1}(r)$.

As $\mathbf{1}+\mathbf{1}_\perp = \operatorname{Spec}\mathcal{F}_1(\mathbb{G})$, by definition of free algebra we have immediately $|\mathcal{A}| = |Hom(\mathcal{F}_1(\mathbb{G}), \mathcal{A})| = |Hom(\operatorname{Spec}\mathcal{A}, \mathbf{1}+\mathbf{1}_\perp)|$ for every $\mathcal{A} \in \mathbb{G}_{fin}$, and by Lemma 6.2, the cardinality of $\mathcal{A}$ is given by the number of subforests of its dual forest, coherently with the fact that the functor Sub associates with a finite forest the Gödel algebra of its subforests.

Lemma 7. *$|Hom(F, \prod_{i=1}^n G_i)| = \prod_{i=1}^n |Hom(F, G_i)|$ for any choice of the finite forests $F, G_1, \ldots, G_n$.*

Proof. Immediate, by the definition of product in categories.

Example 1. We shall compute $N = |\mathcal{E}nd(\mathcal{F}_2(\mathbb{G}))| = |Hom(\mathcal{F}_2(\mathbb{G}), \mathcal{F}_2(\mathbb{G}))|$ using two different methods. The first step is common to the two methods. We use Lemma 4 to determine N as $|Hom(\operatorname{Spec}\mathcal{F}_2(\mathbb{G}), \operatorname{Spec}\mathcal{F}_2(\mathbb{G}))|$.

The first method uses the applicability in this case of Lemma 7 to reduce the computation of N to $|Hom(\operatorname{Spec}\mathcal{F}_2(\mathbb{G}), \operatorname{Spec}\mathcal{F}_1(\mathbb{G}))|^2$, and then, using Lemma 6, to the square of the number of subforests of $\operatorname{Spec}\mathcal{F}_2(\mathbb{G})$.

The prime spectrum of $\mathcal{F}_2(\mathbb{G})$ is the forest $\operatorname{Spec}\mathcal{F}_1(\mathbb{G}) \times \operatorname{Spec}\mathcal{F}_1(\mathbb{G})$, that is, by Theorem 2,

$$(\mathbf{1}+\mathbf{1}_\perp) \times (\mathbf{1}+\mathbf{1}_\perp) \cong (\mathbf{1}\times\mathbf{1}) + (\mathbf{1}\times\mathbf{1}_\perp) + (\mathbf{1}_\perp\times\mathbf{1}) + (\mathbf{1}_\perp\times\mathbf{1}_\perp) \cong$$

$$\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+((\mathbf{1}_\perp\times\mathbf{1})+(\mathbf{1}\times\mathbf{1})+(\mathbf{1}\times\mathbf{1}_\perp))_\perp \cong \mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+(\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp\,.$$

The number of nonempty subforests of $\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+(\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp$ is $(2\cdot3\cdot3)\cdot((2\cdot3\cdot3)+1) = 342$ (an algorithm to compute the number of subforests of a finite forests is presented in [4]). Whence, $|Hom(\mathcal{F}_2(\mathbb{G}), \mathcal{F}_2(\mathbb{G}))| = 342^2 = 116964$.

The second method applies the general recurrence in Theorem 5 to determine N as $|Hom(\operatorname{Spec}\mathcal{F}_2(\mathbb{G}), \operatorname{Spec}\mathcal{F}_2(\mathbb{G}))|$. Writing the forests F and G as in the Theorem, we have $F = G = \operatorname{Spec}\mathcal{F}_2(\mathbb{G}) = \mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+(\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp$, $A = B = \mathbf{1}$, $n = m = 3$, $h_1 = h_2 = k_1 = k_2 = 1$, $h_3 = k_3 = 3$, $T_{11} = T_{21} = T_{31} = U_{11} = U_{21} = U_{31} = \mathbf{1}$, $T_{32} = T_{33} = U_{32} = U_{33} = \mathbf{1}_\perp$. Then

$$\begin{aligned} N = (1+3)^1 \cdot\ & (1+ (1+1)+(1+1)+(1+3))\cdot \\ & (1+ (1+1)+(1+1)+(1+3))\cdot \\ & (1+ [(1+1)\cdot(2+1)\cdot(2+1)]+ \\ & \qquad [(1+1)\cdot(2+1)\cdot(2+1)]+ \\ & \qquad [(1+3)\cdot(4+5)\cdot(4+5)]) = 116964\,, \end{aligned}$$

where, for instance, in the last line, each one of the factors $(4+5)$ arises as $|Hom(\mathbf{1}_\perp, (\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp)|$, that reduces to $|Hom(\mathbf{1}_\perp, ((\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp)^{(2)}| + |Hom(\mathbf{1}_\perp, (\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)|$, with $|Hom(\mathbf{1}_\perp, ((\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp)_\perp)^{(2)}| = |Hom(\mathbf{1}_\perp, (\mathbf{1}+\mathbf{1}+\mathbf{1})_\perp)| = 1+3 = 4$ and $|Hom(\mathbf{1}_\perp, (\mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp))| = 1+2+2 = 5$.

6 Application to related structures

Besides Gödel algebras, there are other varieties of algebras, which constitute the algebraic semantics of some propositional non-classical logics, such that their finite slices are dually equivalent to FF.

Nilpotent minimum algebras are MTL-algebras (see [11] for definitions) further satisfying $\neg(x \,\&\, y) \vee ((x \wedge y) \to (x \,\&\, y)) = 1$ and $\neg\neg x = x$. They form the variety $\mathbb{NM}$. Nilpotent minimum algebras *without negation fixpoint*, that is, those NM-algebras further satisfying $\neg(\neg(x \& x) \& \neg(x \& x)) = (\neg(\neg x \& \neg x)) \& (\neg(\neg x \& \neg x))$ form the variety $\mathbb{NM}^-$. *Idempotent Uninorm Mingle Logic*-algebras, also known as *Bounded Odd Sugihara Monoids*, are idempotent commutative bounded residuated lattices $(A, \wedge, \vee, *, \to, 0, 1, e)$, further satisfying $e \leq (x \to y) \vee (y \to x)$ and $(x \to e) \to e = x$. They form the variety $\mathbb{IUML}$. The following is proved in [9],[2]).

Proposition 1. *$\mathbb{NM}^-_{fin}$ and $\mathbb{IUML}_{fin}$ are dually equivalent to FF . Let us call the functors realising the duality* Spec *again. Then, for every pair $\mathcal{A}, \mathcal{B}$ of finite algebras in $\mathbb{NM}^-$ or in $\mathbb{IUML}$, it holds that $|Hom(\mathcal{A}, \mathcal{B})| = |Hom(\mathrm{Spec}\,\mathcal{B}, \mathrm{Spec}\,\mathcal{A})|$.*

We do not enter details here on the dualities in Prop. 1. We only point out that in the case of $\mathbb{NM}^-_{fin}$ the dual forest to an algebra is order-isomorphic with the poset of its *positive* join-irreducible elements, that is, those greater than their negation, while in the case of $\mathbb{IUML}^-_{fin}$, the dual forest is order-isomorphic with the poset of its *negative* join-irreducible elements, those smaller than their negation. In both cases, knowing or computing the set of join-irreducibles of the involved algebras is sufficient to apply effectively Thm. 5 to compute the cardinality of hom-sets. Analogous considerations apply to all other varieties considered in this section, so we only recall the involved dual equivalences, as their rôle in the applicability of Thm. 5 to compute the number of homomorphisms is the same, *mutatis mutandis*, as in Prop. 1 and Thm. 4.

The 0-free subreducts of Gödel algebras are called *Gödel hoops*, and form the variety $\mathbb{GH}$. The $\{0, 1\}$-free subreducts of IUML-algebras, that is, *Odd Sugihara Monoids*, form the variety $\mathbb{OSM}$.

Proposition 2. *$\mathbb{GH}_{fin}$ and $\mathbb{OSM}_{fin}$ are dually equivalent to FT (see [9,2]).*

Clearly, to compute the number of homomorphisms between two finite algebras either in $\mathbb{GH}_{fin}$ or $\mathbb{OSM}_{fin}$, one just applies Theorem 5 to forests made by a single tree.

Other varieties are dealt with a slight adaptation of our approach. For instance, finite Nilpotent Minimum algebras are dually equivalent to finite *labeled forests*, that is, finite forests where each tree is labeled with a bit $b \in \{0, 1\}$. Arrows are order-preserving open maps that sends the root of a tree labeled b to the root of a tree labeled c, provided that $b \leq c$. The following result is proved in [3].

Proposition 3. *Let FLF be the category of finite labeled forests. Then $\mathbb{NM}_{fin}$ is dually equivalent to FLF.*

We have to slight adapt Theorem 5, in order to apply it to the computation of homomorphisms between finite $\mathbb{NM}$-algebras. Notice that each finite labeled forest $L \in \mathsf{FLF}$ can be displayed as $L = \sum_{i=1}^{n}(T_i, b_i)$, where each T_i is a finite tree and $b_i \in \{0,1\}$ is its label. For $b \in \{0,1\}$ we let $I(b) = \{j \in \{1,\dots,n\} \mid b_j = b\}$ and $L_b = \sum_{j \in I(b)} T_j$.

Theorem 6. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{NM}_{fin}$, and let $F, G \in \mathsf{FLF}$ be their dual finite labeled forests, respectively. Then*

$$|Hom(\mathcal{A},\mathcal{B})| = |Hom(G,F)| = |Hom(G_0, F_0 + F_1)| \cdot |Hom(G_1, F_1)| \, .$$

Proof. By Proposition 3, Lemma 1, and the definition of morphisms in LFL.

Then, to compute $|Hom(\mathcal{A},\mathcal{B})|$, we just apply the recurrence in Theorem 5 to the pairs of finite forests $G_0, F_0 + F_1$ and G_1, F_1.

For any integer $n > 0$, we let $\mathsf{FF}^{(n)}$ be the full subcategory of FF of finite forests of height at most n, that is, those forests $F \in \mathsf{FF}$ such that $F = F^{(n)}$. We let $\mathsf{FT}^{(n)}$ be the full subcategory of $\mathsf{FF}^{(n)}$ whose objects are trees. Further, we define $\mathsf{FLF}^{(n)}$ as the full subcategory of FLF whose objects L are such that $L_0 \in \mathsf{FF}^{(n-1)}$ and $L_1 \in \mathsf{FF}^{(n)}$.

Let $\mathbb{V}$ be any of the varieties considered in this paper. We write $\mathbb{V}^{(n)}$ for the subvariety of $\mathbb{V}$ whose chains have cardinality at most n. Notice that the varieties $\mathbb{V}^{(n)}$ constitute the algebraic semantics of the n-valued versions of the logics having $\mathbb{V}$ as their algebraic semantics.

Proposition 4. *For each integer $n > 0$, the following hold.*

1. $\mathsf{FF}^{(n)}$ *is dually equivalent to* $\mathbb{G}_{fin}^{(n+1)}$, $\mathbb{NM}_{fin}^{-(2n)}$, $\mathbb{IUML}_{fin}^{(2n+1)}$.
2. $\mathsf{FT}^{(n)}$ *is dually equivalent to* $\mathbb{GH}_{fin}^{(n)}$, $\mathbb{OSM}_{fin}^{(2n-1)}$.
3. $\mathsf{FLF}^{(n)}$ *is dually equivalent to* $\mathbb{NM}_{fin}^{(2n)}$.

Also in these cases, the dualities stated in Proposition 4 allows us to use Theorem 5 to compute the number of homomorphisms between any two finite algebras belonging to the above mentioned varieties.

7 Conclusion

We observe that the cardinality of the monoid of endomorphisms of a finite non-trivial[4] Boolean algebra $\mathcal{B}$ completely determines $\mathcal{B}$. As a matter of fact $|\mathcal{E}nd(\mathcal{B})| = n^n$ where n is the number of atoms of $\mathcal{B}$. Notice that if $n^n = m^m$ for n and m positive integers, then $n = m$. Whence, $\mathcal{B}$ is the unique Boolean algebra of 2^n elements. This property does not generalise to Gödel algebras, as the following example shows.

[4] The trivial, one-element algebra has clearly just the identity as endomorphism, as it is the case for the standard two-element Boolean algebra.

Example 2. Let $\mathcal{A} = \{0,1\} \times \{0,1\}$ be the direct product of two copies of the two-element Boolean algebra, and $\mathcal{B}$ be the four-element Gödel chain. Then $\operatorname{Spec} \mathcal{A} \cong \mathbf{1} + \mathbf{1}$, while $\operatorname{Spec} \mathcal{B} \cong (\mathbf{1}_\perp)_\perp$. Whence, using Theorem 4 and Theorem 5, we find that $|\mathcal{E}nd(\mathcal{A})| = 4 = |\mathcal{E}nd(\mathcal{B})|$. On the other hand, the two monoids $\mathcal{E}nd(\mathcal{A})$ and $\mathcal{E}nd(\mathcal{B})$ are clearly not isomorphic, as the former contains as submonoid the group $\mathcal{A}ut(\mathcal{A}) \cong \mathbb{Z}_2$, while the latter does not, as $\mathcal{A}ut(\mathcal{B})$ is the trivial one-element group (it contains just the identity map).

Example 2 suggests the investigation of the following problems:

- Does the monoid structure of $\mathcal{E}nd(\mathcal{A})$ completely determine the finite Gödel algebra $\mathcal{A}$?
- Does the pair of natural numbers $(|\mathcal{E}nd(\mathcal{A})|, |\mathcal{A}ut(\mathcal{A})|)$ completely determine the finite Gödel algebra $\mathcal{A}$?

References

1. S. Aguzzoli. Automorphism groups of Lindenbaum algebras of some propositional many-valued logics with locally finite algebraic semantics. *Proceedings of FUZZ-IEEE 2020*, pp. 1–8, 2020.
2. S. Aguzzoli, S. Bova, B. Gerla. Free Algebras and Functional Representation for Fuzzy Logics. Chapter IX of Handbook of Mathematical Fuzzy Logic - Volume 2. P. Cintula, P. Hájek, C. Noguera Eds., Studies in Logic, **38**, College Publications, London, pp. 713–791, 2011.
3. S. Aguzzoli, M. Busaniche, V. Marra. Spectral duality for finitely generated nilpotent minimum algebras, with applications. Journal of Logic and Computation, **17**, 749–765, 2007.
4. S. Aguzzoli, P. Codara. Recursive formulas to compute coproducts of finite Gödel algebras and related structures. Proc. of FUZZ-IEEE 2016. pp. 201–208, 2016.
5. S. Aguzzoli, P. Codara. Towards an Algebraic Topos Semantics for Three-valued Gödel Logic. Proceedings of FUZZ-IEEE 2021, pp. 1-8, 2021.
6. S. Aguzzoli, B. Gerla. Automorphism Groups of Finite BL-Algebras. Proceedings of IPMU 2020, pp. 666–679, 2020.
7. S. Aguzzoli, B. Gerla. Averaging the truth value of formulas in Gödel logic. Proceedings of FUZZ-IEEE 2023, pp. 1-8, 2023.
8. S. Aguzzoli, B. Gerla, V. Marra. The automophism group of finite Gödel algebras. Proc. ISMVL 2010. IEEE Computer Society Press, 21–26, 2010.
9. S. Aguzzoli, T. Flaminio, E. Marchioni. Finite forests, their algebras and logics, manuscript.
10. O. M. D'Antona and V. Marra. Computing coproducts of finitely presented Gödel algebras. *Ann. Pure Appl. Logic*, 142(1-3):202–211, 2006.
11. F. Esteva, L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. Fuzzy Sets and Systems, **124**, 271–288, 2001.
12. P. Hájek. *Metamathematics of fuzzy logic.* Kluwer Academic Publishers, 1998.
13. A. Horn. Logic with truth values in a linearly ordered Heyting algebra. *J. Symbolic Logic*, 34:395–408, 1969.
14. A. Horn. Free L-algebras. *J. Symbolic Logic*, 34:475–480, 1969.
15. P. T. Johnstone. *Stone spaces.* Cambridge University Press, Cambridge, 1982.
16. S. MacLane. *Categories for the working mathematician.* Springer, 1971.