A new duality concept for AC and interaction operators through the IE-integral

Katsushige Fujimoto[1] and Aoi Honda[2]

[1] Fukushima University, 1 Kanayagawa, Fukushima 960-1296, JAPAN
fujimoto@sss.fukushima-u.ac.jp
[2] Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, JAPAN
aoi@ai.kyutech.ac.jp

Abstract. This paper explores interaction operators utilized within the framework of the inclusion and exclusion integral (IE-integral). Specifically, we introduce a novel duality concept for interaction operators and the associative and commutative binary operators (AC-operators) that produce them. These concepts are derived from the conventional duality of set functions through the framework of IE-integrals. Moreover, we reframe IE-integrals and the aforementioned new concepts from the standpoint of discrete derivatives.

Keywords: duality, monotone measure, inclusion-exclusion integral, Möbius transform, interaction operator, discrete derivative, AC-operator

1 Introduction

The inclusion and exclusion integral (IE-integral) constitutes a class of integrals based on monotone measures. The definition of this integral involves not only monotone measures but also interaction operators, providing flexibility in selecting operations and thereby introducing integrals with various properties. For example, choosing the min-operator as the interaction operator yields the Choquet integral, demonstrating that appropriate choices of operators enable the definition of integrals with desirable properties. However unresolved issues persist regarding the properties of these interaction operators and IE-integrals.

In this paper, we focus on interaction operators derived from associative and commutative binary operators (AC-operators), offering fresh theoretical insights into IE-integrals and their interaction operators by introducing a novel duality concept through the IE-integral framework.

The structure of this paper is as follows: In Section 2, we introduce concepts related to interaction operations and the definition of IE-integral. Section 3 presents the new duality concept for interaction operators and the AC-operators that generate them. We also discuss pairs of dual operators, the uniqueness of the arithmetic mean as self-dual interaction operations, and the relationship between the IE-integral and the Shapley values. Section 4 introduces two types of discrete derivatives, demonstrating that computations related to the IE-integral correspond to the high-order differentials of discrete functions.

2 Preliminaries

Let $X = \{1, \ldots, n\}$ be a finite set. We denote the cardinality of a subset $A \subseteq X$ as $|A|$, and the power set of X as 2^X.

Definition 1 (monotone measure). *A set function $\mu : 2^X \to [0,1]$ is said to be a monotone measure if μ satisfies the following two conditions:*

1. $\mu(\emptyset) = 0$, *and*
2. $\mu(A) \le \mu(B)$ *whenever* $A \subseteq B, A, B \in 2^X$.

Let $\circledast$ be an associative and commutative binary operator (also abbreviated as AC-operator) on $[0,1]^2$. The iterations of the operations $\circledast$ is denoted as $\underset{i\in\{1,\cdots,n\}}{\circledast} x_i$.

Definition 2 (t-norm and t-conorm [5]). *An AC-operator $\circledast$ on $[0,1]^2$ is said to be* t-norm (*resp.* t-conorm) *if the following conditions 1 and 2 (resp. 1 and 3).*

1. $x \circledast z \le y \circledast z$ *if* $x \le y \quad \forall x, y, z \in [0,1]$
2. $x \circledast 1 = x \quad \forall x \in [0,1]$
3. $x \circledast 0 = x \quad \forall x \in [0,1]$

Definition 3 (interaction operator [4]). *Let $f : X \to [0,1]$ be a function on X, and $\circledast$ be an AC-operator on $[0,1]^2$. An interaction operator $I_f : 2^X \to [0,1]$ on X with respect to f is a set function on 2^X satisfying the following conditions 1 and 2. A* conjunctive (resp. disjunctive) interaction operator *I_f with respect to f is a set function on 2^X satisfying the following conditions 2 and 3 (resp. 2 and 4). An interaction operator induced by AC-operator $\circledast$ is an interaction operator $I_f^{\circledast} : 2^X \to [0,1]$ on X satisfying the following condition 5:*

1. $I_f(\emptyset) \in \{0,1\}$,
2. $I_f(\{i\}) = f(i) \quad \forall i \in X$,
3. $I_f(\emptyset) = 1$ *and* $I_f(S) \ge I_f(T)$ *whenever* $S \subseteq T$,
4. $I_f(\emptyset) = 0$ *and* $I_f(S) \le I_f(T)$ *whenever* $S \subseteq T$,
5. $I_f^{\circledast}(S) = \underset{i\in S}{\circledast} f(i) \quad \forall S \in 2^X \ (|S| > 1)$.

Example 1. [4] Let $\otimes$ be a t-norm on $[0,1]^2$. A set function I_f on 2^X with respect to a function $f : X \to [0,1]$ defined by

$$I_f(S) := \begin{cases} 1 & \text{if } S = \emptyset, \\ f(i) & \text{if } S = \{i\} \text{ for some } i \in X, \\ \bigotimes_{i\in S} f(i) & \text{otherwise} \end{cases}$$

is a conjunctive interaction operator. Let $\oplus$ be a t-conorm on $[0,1]^2$. A set function I_f on 2^X with respect to a function $f : X \to [0,1]$ defined by

$$I_f(S) := \begin{cases} 0 & \text{if } S = \emptyset, \\ f(i) & \text{if } S = \{i\} \text{ for some } i \in X, \\ \bigoplus_{i\in S} f(i) & \text{otherwise} \end{cases}$$

is a disjunctive interaction operator.

Definition 4 (inclusion-exclusion integral [4]). *Let $f : X \to [0,1]$ be a function on X and $\mu : 2^X \to [0,1]$ a monotone measure. The* inclusion-exclusion integral *of f with respect to an interaction operator I_f with respect to f and μ, denoted by $(IE)\int f \, d\mu$, is defined by*

$$(IE)\int f \, d\mu := \sum_{S \in 2^X} I_f(S) \cdot m_\mu(S),$$

where m_μ is the Möbius transform of μ, i.e., $m_\mu(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mu(T)$ for $S \in X$.

It has been known that the inclusion-exclusion integral can be represented in another form as follows:

$$(IE)\int f \, d\mu = \sum_{S \in 2^X} M_f^I(S) \cdot \mu(S),$$

where

$$M_f^I(S) := \sum_{T \supseteq S} (-1)^{|T \setminus S|} I_f(T) = \sum_{T \subseteq S^c} (-1)^{|T|} I_f(S \cup T) \quad \text{for } S \in 2^X. \tag{1}$$

Inversely,

$$I_f(S) = \sum_{T \supseteq S} M_f^I(T).$$

Proposition 1. [4]

$$f(i) = \sum_{S \ni i} M_f^I(S) \quad \forall f : X \to [0,1],$$

i.e.,

$$f = \sum_{S \in 2^X} M_f^I(S) \cdot \chi_S,$$

where χ_S is the characteristic function of S.

Example 2. The inclusion-exclusion integral of f with respect to the interaction operator induced by the min-operator $\wedge$ and μ coincides with the Choquet integral of f with respect to μ, i.e.,

$$(C)\int f \, d\mu = \sum_{S \in 2^X} \bigwedge_{i \in S} f(i) \cdot m_\mu(S).$$

In [1], integral with respect to non-additive measures, including Choquet and Sugeno integrals, is conceptualized within a more generalized framework termed *Choquet-Sugeno-like operator*. The inclusion-exclusion integral is also considered as one of the Choquet-Sugeno-like operators within this framework, where interaction operators constitute one class of *conditional aggregation operators.*

3 Duality

In this section, we introduce a new duality concept of AC and interaction operators, which is derived from the duality of non-additive measures through the IE-integral.

Definition 5 (duality of monotone measures [6]). *Two monotone measures μ and μ^* are said to be dual if*

$$\mu^*(S) = \mu(X) - \mu(S^c) \quad \forall S \in 2^X.$$

Example 3. The belief function $Bel : 2^X \to [0,1]$ and the plausibility function $Pl : 2^X \to [0,1]$ in the evidence theory[9] are dual. Besides, every additive measure $v : 2^X \to [0,1]$ is self-dual, i.e., $v(S) = v(X) - v(S^c)$ for any $S \in 2^X$.

Definition 6 (duality of AC-operators [7]). *Two AC-operators $\otimes$ and $\oplus$ are said to be dual if*

$$a \oplus b = 1 - (1-a) \otimes (1-b) \quad \forall a, b \in [0,1],$$

i.e.,

$$\bigoplus_i a_i = 1 - \bigotimes_i (1 - a_i) \quad \forall \{a_i\} \subseteq [0,1]$$

Example 4. All of the following pairs of AC-operators (pairs of t-norm and t-conorm) are dual, respectively: (min $\wedge$, max $\vee$), (algebraic product $\bullet$, algebraic sum $\boxplus$), (bounded product $\odot$, bounded sum $\oplus$), and (drastic product $\top$, drastic sum $\perp$), where $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$, $a \bullet b = ab$, $a \boxplus b = 1 - (1-a)(1-b)$, $a \odot b = \max\{0, a+b-1\}$, $a \oplus b = \min\{1, a+b\}$,

$$a \top b = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a \perp b = \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 7 (duality of interaction operators). *Two interaction operators I_f and I_f^* with respect to f are said to be dual in IE-integral if*

$$\sum_{S \in 2^X} I_f(S) \cdot m_\mu(S) = \sum_{S \in 2^X} I_f^*(S) \cdot m_{\mu^*}(S) \quad \forall f : X \to [0,1], \forall \mu : 2^X \to [0,1].$$

Let $I_f^{\otimes}$ be the interaction operator induced by an AC-operator $\otimes$ and $I_f^{\oplus}$ the interaction operator induced by an AC-operator $\oplus$. The AC-operators $\otimes$ and $\oplus$ are said to be dual in IE-integral if $I_f^{\otimes}$ and $I_f^{\oplus}$ are dual in IE-integral.

In other words, the duality between AC and interaction operators is established through the duality of monotone measures within the context of the IE-integral.

Proposition 2. *The interaction operator* $I_f^{\wedge}$ *induced by the* min*-operator* ($\wedge$) *and the interaction operator* $I_f^{\vee}$ *induced by the* max*-operator* ($\vee$) *are dual in IE-integral, i.e., The AC-operators* $\wedge$ *and* $\vee$ *are dual in IE-integral.*

Proposition 3. *Let* I_f *and* I_f^* *be two interaction operators on* X *and* $M_f^I(S) := \sum_{T \supset S}(-1)^{|T \setminus S|} I_f(T)$. *Then* I_f *and* I_f^* *are dual in IE-integral if and only if the following two conditions hold: for any* $f : X \to [0,1]$,

1. $\displaystyle\sum_{\emptyset \neq S \in 2^X} M_f^{I^*}(S) = M_f^I(X)$,
2. $M_f^{I^*}(S^c) = -M_f^I(S) \quad \forall S \in 2^X \setminus \{\emptyset, X\}$.

Proposition 4. *The interaction operator* $I_f^{\bullet}$ *induced by the algebraic product* $\bullet$ *and the interaction operator* $I_f^{\boxplus}$ *induced by the algebraic sum* $\boxplus$ *are dual in IE-integral, i.e., The AC-operators* $\bullet$ *and* $\boxplus$ *are dual in IE-integral.*

Example 5. The pairs ($\odot$, $\oplus$) and ($\top$, $\perp$) are dual AC-operators. However, they are not dual in IE-integral. Indeed, for a function $f : \{1,2,3\} \to [0,1]$ such as $f(1) = 0.7, f(2) = 0.5, f(3) = 0.3$, $M_f^{\odot}(\{1,2\}) = 0.2$ and $M_f^{\oplus}(\{3\}) = -0.5$, and $M_f^{\top}(\{1,2\}) = 0$ and $M_f^{\perp}(\{3\}) = -0.7$. These violate condition 2 in Proposition 3.

Definition 8. *For a function* $f : X \to [0,1]$, *the interaction operator induced by the arithmetic mean* I_f^{am} *with respect to* f *is defined by*

$$I_f^{am}(S) := \frac{\sum_{i \in S} f(i)}{|S|}.$$

Applying Proposition 3 to I_f^{am} yields the following proposition.

Proposition 5. *The interaction operator induced by the arithmetic mean is self-dual, i.e.,*

$$\sum_{S \in 2^X} I_f^{am}(S) \cdot m_\mu(S) = \sum_{S \in 2^X} I_f^{am}(S) \cdot m_{\mu^*}(S),$$

for any function $f : X \to [0,1]$ *and any monotone measure* $\mu : 2^X \to [0,1]$.

This proposition can also be proved from the following Propositions 6 and 7.

Definition 9 (The Shapley value [8]). *Let* $\mu : 2^X \to [0,1]$ *be a monotone measure. The Shapley value* $\Phi(\mu) = (\phi_1(\mu), \cdots, \phi_n(\mu))$ *of* μ *is defined by*

$$\phi_i(\mu) = \sum_{\substack{S \in 2^X \\ S \not\ni i}} \frac{|S|!\ (n - |S| - 1)!}{n!} \left(\mu(S \cup \{i\}) - \mu(S)\right), \quad i \in X$$

Proposition 6. *[8] Let* $\mu : 2^X \to [0,1]$ *be a monotone measure on* 2^X *and* $\mu^* : 2^X \to [0,1]$ *be its dual (i.e.,* $\mu^*(S) = \mu(X) - \mu(S^c)$ *for* $S \in 2^X$*). Then,*

$$\Phi(\mu) = \Phi(\mu^*),$$

where $\Phi(\mu)$ *is the Shapley value of* μ. *This property is recognized as the self-duality property of the Shapley value.*

Proposition 7. *Let I^{am} be the interaction operator induced by arithmetic mean (see, Definition 8), and $\mu : 2^X \to [0,1]$ a monotone measure on 2^X. The inclusion-exclusion integral of $f : X \to [0,1]$ with respect to I^{am} and μ can be represented as follows:*

$$(IE)\int f\ d\mu = \int f\ d\ \phi_i(\mu) = \sum_{i\in X} f(i)\cdot\phi_i(\mu),$$

where $\Phi(\mu) := (\phi_1(\mu),\ldots,\phi_n(\mu))$ is the Shapley value of μ.

Corollary 1. *Let $f^j := \chi_{\{j\}}$ and $I := I^{am}$. Then we have*

$$(IE)\int f^j\ d\mu = \phi_j(\mu).$$

Proposition 8. *Let I^{am} be the interaction operator induced by the arithmetic mean (see, Definition 8), and $\mu : 2^X \to [0,1]$ a monotone measure on 2^X. The inclusion-exclusion integral of $f : X \to [0,1]$ with respect to I^{am} and μ is an integral functional, that is,*

1. *$f = 0$ implies $(IE)\int f\ d\mu = 0$,*
2. *$f \le g$, that is, $f_i \le g_i, i \in X$ implies $(IE)\int f\ d\mu \le (IE)\int g\ d\mu$.*

4 Discrete derivative of set functions

In this section, we discuss the IE-integral and its duality from the standpoint of discrete derivatives.

Definition 10 (discrete derivative). *Given a set function $\mu : 2^X \to [0,1]$ and two disjoint non-empty subsets $S, T \subseteq X$, we denote by $\Delta_S^{\mathrm{FW}}\mu(T)$ the forward-S-derivative (left-hand S-derivative) of μ at T [3], which is recursively defined by*

$$\Delta_{\{i\}}^{\mathrm{FW}}\mu(T) := \mu(T\cup\{i\}) - \mu(T) \quad \forall i \in S,$$

and

$$\Delta_S^{\mathrm{FW}}\mu(T) := \Delta_{\{i\}}^{\mathrm{FW}}[\Delta_{S\setminus\{i\}}^{\mathrm{FW}}\mu(T)] \quad \forall i \in S.$$

On the other hand, the backward-S-derivative (right-hand S-derivative) of μ at $S\cup T$, denoted by $\Delta_S^{\mathrm{BW}}\mu(S\cup T)$, is recursively defined by

$$\Delta_{\{i\}}^{\mathrm{BW}}\mu(S\cup T) := \mu(S\cup T\setminus\{i\}) - \mu(S\cup T) \quad \forall i \in S,$$

and

$$\Delta_S^{\mathrm{BW}}\mu(S\cup T) := \Delta_{\{i\}}^{\mathrm{BW}}[\Delta_{S\setminus\{i\}}^{\mathrm{BW}}\mu(S\cup T\setminus\{i\})] \quad \forall i \in S.$$

We can easily prove, in a similar way as in [3], by induction on $|S|$ that

$$\Delta_S^{\mathrm{FW}}\mu(T) = \sum_{L\subseteq S} (-1)^{|S\setminus L|}\mu(T\cup L)$$

and

$$\Delta_S^{\mathrm{BW}}\mu(S\cup T) = \sum_{L\subseteq S} (-1)^{|L|}\mu(T\cup L).$$

Example 6. Let us consider discrete $\{i,j\}$-derivatives of μ at T or $T\cup\{i,j\}$ through Figure 4.1. For a $T\in 2^X$ and $i,j\in X\setminus T$,

$$\Delta_{\{i\}}^{\mathrm{FW}}\mu(T) = \mu(T\cup\{i\}) - \mu(T)$$

and

$$\Delta_{\{i\}}^{\mathrm{FW}}\mu(T\cup\{j\}) = \mu(T\cup\{i,j\}) - \mu(T\cup\{j\})$$

(red arrows in Figure 4.1 (a)). Then,

$$\begin{aligned}\Delta_{\{i,j\}}^{\mathrm{FW}}\mu(T) &= \Delta_{\{i\}}^{\mathrm{FW}}\mu(T\cup\{j\}) - \Delta_{\{i\}}^{\mathrm{FW}}\mu(T)\\ &= \mu(T\cup\{i,j\}) - \mu(T\cup\{i\}) - \mu(T\cup\{j\}) + \mu(T)\end{aligned}$$

(the blue arrow in Figure 4.1 (a)). While,

$$\Delta_{\{i\}}^{\mathrm{BW}}\mu(T\cup\{i\}) = \mu(T) - \mu(T\cup\{i\})$$

and

$$\Delta_{\{i\}}^{\mathrm{BW}}\mu(T\cup\{i,j\}) = \mu(T\cup\{j\}) - \mu(T\cup\{i,j\})$$

(red arrows in Figure 4.1 (b)). Then,

$$\begin{aligned}\Delta_{\{i,j\}}^{\mathrm{BW}}\mu(T\cup\{i,j\}) &= \Delta_{\{i\}}^{\mathrm{BW}}\mu(T\cup\{i\}) - \Delta_{\{i\}}^{\mathrm{BW}}\mu(T\cup\{i,j\})\\ &= \mu(T\cup\{i,j\}) - \mu(T\cup\{i\}) - \mu(T\cup\{j\}) + \mu(T)\end{aligned}$$

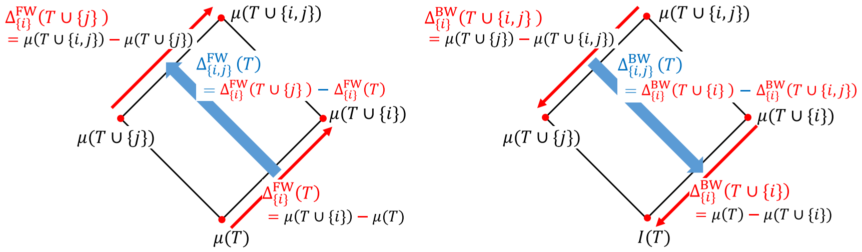
(the blue arrow in Figure 4.1 (b)).

Example 7. Let us consider discrete $\{i,j,k\}$-derivatives of μ at T or $T\cup\{i,j,k\}$ through Figure 4.2. For a $T\in 2^X$ and $i,j,k\in X\setminus T$, similar to Example 6,

$$\begin{aligned}\Delta_{\{i,j\}}^{\mathrm{FW}}\mu(T) &= \Delta_{\{i\}}^{\mathrm{FW}}\mu(T\cup\{j\}) - \Delta_{\{i\}}^{\mathrm{FW}}\mu(T)\\ &= [\mu(T\cup\{i,j\}) - \mu(T\cup\{i\})] - [\mu(T\cup\{j\}) - \mu(T)]\end{aligned}$$

and

$$\begin{aligned}\Delta_{\{i,k\}}^{\mathrm{FW}}\mu(T\cup\{j\}) &= \Delta_{\{i\}}^{\mathrm{FW}}\mu(T\cup\{j,k\}) - \Delta_{\{i\}}^{\mathrm{FW}}\mu(T\cup\{j\})\\ &= [\mu(T\cup\{i,j,k\}) - \mu(T\cup\{i,j\})] - [\mu(T\cup\{j,k\}) - \mu(T\cup\{j\})]\end{aligned}$$



(a) forward $\{i,j\}$-derivative at T (b) backward $\{i,j\}$-derivative at $T \cup \{i,j\}$

Fig. 4.1. $\{i,j\}$-derivative at T (or $T \cup \{i,j\}$)

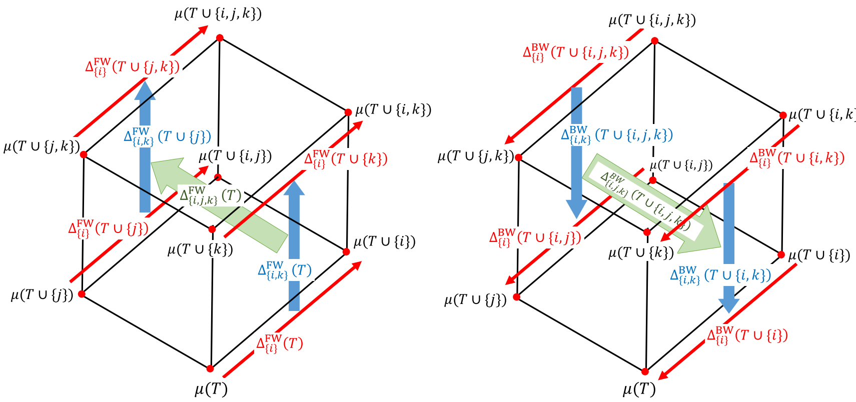
(blue arrows in Figure 4.2 (c)). Then,

$$\begin{aligned}\Delta^{\mathrm{FW}}_{\{i,j,k\}}\mu(T) &= \Delta^{\mathrm{FW}}_{\{i,k\}}\mu(T \cup \{j\}) - \Delta^{\mathrm{FW}}_{\{i,k\}}\mu(T) \\ &= [\mu(T \cup \{i,j,k\}) - \mu(T \cup \{i,j\}) - \mu(T \cup \{j,k\}) + \mu(T \cup \{j\})] \\ &\qquad - [\mu(T \cup \{i,k\}) - \mu(T \cup \{i\}) - \mu(T \cup \{k\}) + \mu(T)]\end{aligned}$$

(the green arrow in Figure 4.2 (c)). While,

$$\begin{aligned}\Delta^{\mathrm{BW}}_{\{i,j,k\}}\mu(T \cup \{i,j,k\}) &= -\mu(T \cup \{i,j,k\}) + \mu(T \cup \{i,j\}) + \mu(T \cup \{j,k\}) - \mu(T \cup \{j\}) \\ &\qquad + \mu(T \cup \{i,k\}) - \mu(T \cup \{i\}) - \mu(T \cup \{k\}) + \mu(T)\end{aligned}$$

(the green arrow in Figure 4.2 (d)).



(c) forward $\{i,j,k\}$-derivative at T (d) backward $\{i,j,k\}$-derivative at $T \cup \{i,j,k\}$

Fig. 4.2. $\{i,j,k\}$-derivative at T (or $T \cup \{i,j,k\}$)

Proposition 9. *Let I_f be an interaction operator with respect to $f : X \to [0,1]$ and M_f^I a set function induced by the equation (1). Then, it holds that*

$$M_f^I(S) = \Delta_{S^c}^{\mathrm{BW}} I_f(S \cup S^c) = \Delta_{S^c}^{\mathrm{BW}} I_f(X),$$

$$m_\mu(S) = \sum_{L \subseteq S} (-1)^{|S \setminus L|}(\emptyset \cup L) = \Delta_S^{\mathrm{FW}} \mu(\emptyset).$$

Corollary 2. *Let $f : X \to [0,1]$ be a function, $I_f : 2^X \to [0,1]$ an interaction operator with respect to f, and $\mu : 2^X \to [0,1]$ a monotone measure. Then, the inclusion and exclusion integral of f with respect to I_f and μ is represented, through the use of discrete derivatives, as*

$$(IE) \int f \, d\mu = \sum_{S \in 2^X} I_f(S) \cdot \Delta_S^{\mathrm{FW}} \mu(\emptyset) = \sum_{S \in 2^X} \Delta_{S^c}^{\mathrm{BW}} I_f(X) \cdot \mu(S).$$

In the preceding section, we explored the duality between pairs of dual AC-operators ($\wedge$, $\vee$) and ($\bullet$, $\boxplus$) exhibited duality in the context of the IE-integral, and pairs of dual AC-operators ($\odot$, $\oplus$) and ($\top$, $\perp$) that did not. Hereafter, we will analyze the distinctions between these pairs utilizing the concept of discrete derivatives.

Proposition 10. *Let $I_f^\wedge$ be the interaction operator induced by the* min*-operator $\wedge$. Then, for any function $f : X \to [0,1]$,*

$$\Delta_S^{\mathrm{BW}} I_f^\wedge(S \cup T) \geq 0 \ \forall S, T \in 2^X, \ S \cap T = \emptyset$$

Proposition 11. *Let $I_f^\bullet$ be the interaction operator induced by the algebraic product $\bullet$. Then, for any function $f : X \to [0,1]$,*

$$\Delta_S^{\mathrm{BW}} I_f^\bullet(S \cup T) \geq 0 \ \forall S, T \in 2^X, \ S \cap T = \emptyset$$

Corollary 3.

$$M_f^\wedge(S) \in [0,1] \quad \forall S (\neq \emptyset) \in 2^X,$$

$$M_f^\bullet(S) \in [0,1] \quad \forall S (\neq \emptyset) \in 2^X,$$

i.e., Any function $f : X \to [0,1]$ on X can be extended to functions $M_f^\wedge$, $M_f^\bullet : 2^X \to [0,1]$ on 2^X through the use of interaction operators $\wedge$ and $\bullet$ as

$$f = \sum_{S \in 2^X} M_f^\wedge(S) \cdot \chi_S = \sum_{S \in 2^X} M_f^\bullet(S) \cdot \chi_S.$$

Example 8. $M_f^\top$ is not always a function on 2^X to $[0,1]$. Indeed, for a function $f : \{1,2,3\} \to [0,1]$ such as $f(1) = 1, f(2) = 0.8, f(3) = 0.6$, then $M_f^\top(\{1\}) = -0.4 \notin [0,1]$. Besides, neither is $M_f^\odot$, for a function $f : \{1,2,3,4\} \to [0,1]$ such as $f(1) = 1, f(2) = 0.8, f(3) = 0.6, f(4) = 0.2$, then $M_f^\odot(\{1\}) = -0.2 \notin [0,1]$.

Proposition 12. *Let I and I^* be dual (in IE-integral) interaction operators on X. If $\Delta_S^{\mathrm{BW}} I(S \cup T) \geq 0$ for any disjoint non-empty disjoint subsets $S, T \in 2^X$, then $\Delta_S^{\mathrm{BW}} I^*(S \cup T) \leq 0$ for any non-empty disjoint subsets $S, T \in 2^X$.*

From Proposition 2, 4, 10, 11, and 12, we have the following proposition and its corollary:

Proposition 13. *Let $I_f^{\vee}$ be the interaction operator induced by the max-operator $\vee$ and $I_f^{\boxplus}$ the interaction operator induced by the algebraic sum operator $\boxplus$. Then,*

$$\Delta_S^{\mathrm{BW}} I_f^{\vee}(S \cup T) \leq 0, \quad \text{and} \quad \Delta_S^{\mathrm{BW}} I_f^{\boxplus}(S \cup T) \leq 0$$

for any disjoint non-empty sets $S, T \in 2^X$.

Corollary 4. *Let $I_f^{\vee}$ be the interaction operator induced by the max-operator $\vee$ and $I_f^{\boxplus}$ the interaction operator induced by the algebraic sum operator $\boxplus$. Then,*

$$M_f^{\vee}(S) \leq 0, \quad \text{and} \quad M_f^{\boxplus}(S) \leq 0$$

for any non-empty subset $S(\neq X) \in 2^X$ and any function $f : X \to [0, 1]$.

5 Conclusion

In this study, we introduced a new duality concept for AC-operators. This concept is derived from the duality of monotone measures through the IE-integral and is referred to as the duality in IE-integral. As you may know, t-norms and t-conorms are one of the AC-operators. AC-operators also have a duality concept as stated in Definition 6. Dual AC-operators, such as max and min, are also dual in IE-integral, so are the algebraic sum and product. However, not all AC-operators that are dual are necessarily dual in IE-integral. Additionally, we reframe IE-integrals from the perspective of discrete derivatives. Furthermore, from a different perspective, certain relationships between the interaction operator and the conditional aggregation operator introduced in [1] have been acknowledged. Clarifying these relationships is a key task for future work.

Acknowledgment This work was supported by JSPS KAKENHI Grant Number 21K04525.

References

1. M.Boczek, O.Hutn?k, and M.Kaluszka, "Choquet-Sugeno-like operator based on relation and conditional aggregation operators," Information Sciences **582**, 1-21 (2022).
2. M.Grabisch, "k-order additive discrete fuzzy measures and their representation," Fuzzy sets and systems **92**, 167-189 (1997).

3. M.Grabisch, J.-L.Marichal, and M.Roubens, “Equivalent representations of set functions,” Mathematics of Operations Research **25**(2), 157-178 (2000).
4. A.Honda and Y.Okazaki, “Theory of inclusion-exclusion integral,” Information Sciences **376**, 136-147 (2017).
5. E.Klement, R.Mesiar, and E.Pap, Triangular Norms (Kluwer Academic Publishers, 2000).
6. X.Liu, “Study on the dual problems in the theory of fuzzy measure and fuzzy integral,” The Journal of fuzzy mathematics **2**, 555?563 (1994).
7. G.Mayor and J.Torrens, “Duality for a class of binary operations on [0,1],” Fuzzy Sets and Systems **47**, 77-80 (1992).
8. M.Oishi and et al., “Duality and anti-duality in TU games applied to solutions, axioms, and axiomatizations,” Journal of Mathematical Economics **63**, 44-53 (2016).
9. G.Shafer, A Mathematical Theory of Evidence (Princeton University Press, 1976).