Functors in Fuzzy Category Theory

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Abstract. Fuzzy category theory describes category-like structures in which potential objects and potential morphisms are respectively objects and morphisms only to a certain degree. This also allows us to look at categories with crisp objects and morphisms from the point of fuzzy set theory without necessarily framing, explicitly or implicitly, the objects or morphisms of a category in a fuzzy way. Building on the work done on functors in fuzzy categories, we further develop the theory until the notion of adjoint functors using the unit and co-unit. We also provide a sufficient condition for a fuzzy functor so that it admits a left adjoint with a certain degree.

Keywords: Fuzzy categories · Adjoint functors · Natural transformations

1 Introduction

Since the inception of fuzzy set theory, significant efforts have been made to find fuzzy analogues to fundamental concepts in classical mathematics. Several approaches to fuzzy topology have been developed [1,3,9], and these approaches have even been compared with the tools of category theory [7, 8], resulting in the creation of categories whose objects are fuzzy topologies. Similarly, fuzzy algebraic structures have been developed [5], and fuzzy topologies on algebraic structures have also been explored [6]. Although we are discussing fuzzy structures, the underlying categories remain crisp. Naturally, there is also an interest in developing a fuzzy analogue of one of the most influential mathematical paradigm shifts — category theory.

Instead of researching classical categories from the perspective of fuzzy structures, the goal of this paper is to develop fuzzy category theory. In other words, we aim to develop a theory of structures that resemble categories, but where potential objects and potential morphisms are respectively objects and morphisms only to some degree [10]. Just as fuzzy set theory encompasses the concept of a classical set, fuzzy category theory includes classical categories as a special case. By additionally keeping track of the membership degrees of objects and morphisms, we can locally describe both fuzzy and crisp structures in an otherwise context-free framework. With the help of fuzzy categories it is also possible to model situations within crisp structures by relaxing the requirements on the objects or morphisms, e.g., we can assign a measure of continuity to arbitrary set morphisms to extend the class of morphisms for the category of topological spaces **Top**.

Fuzzy category theory has seen relatively little progress. While various examples of fuzzy categories have been constructed, the theory itself has not seen development beyond the definitions of fuzzy morphisms and functors [12]. In this paper, we take the next step by introducing a definition for natural transformations in fuzzy categories, as well as defining adjoint functors through the unit and co-unit whilst also taking the first steps into proving an analogue of the adjoint functor theorem for functors in fuzzy categories.

2 Fuzzy Category Theory

2.1 Preliminaries

Definition 1 ([2]). *A GL-monoid is a complete lattice, whose universal upper and lower bounds are respectively $\top$ and $\bot$, enriched with a further binary operation $*$, i.e., a triple $(L, \leq, *)$ such that:*

1. *$*$ is monotone, i.e. $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$ for any $\alpha, \beta, \gamma \in L$;*
2. *$*$ is commutative, i.e. $\alpha * \beta = \beta * \alpha$ for any $\alpha, \beta \in L$;*
3. *$*$ is associative, i.e. $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ for any $\alpha, \beta, \gamma \in L$;*
4. *$(L, \leq, *)$ is integral, i.e. $\top$ acts as the unity: $\alpha * \top = \alpha$ for any $\alpha \in L$;*
5. *$\bot$ acts as the zero element in $(L, \leq, *)$, i.e., $\alpha * \bot = \bot$ for any $\alpha \in L$;*
6. *$*$ is distributive over arbitrary joins, i.e. $a * \left(\bigvee_j \beta_j\right) = \bigvee_j (a * \beta_j)$ for any $\alpha \in L$ and $\{\beta_j \mid j \in J\} \subseteq L$;*
7. *$(L, \leq, *)$ is divisible, i.e. any $\alpha, \beta \in L$ for which $\alpha \leq \beta$ implies the existence of $\gamma \in L$ such that $\alpha = \beta * \gamma$.*

Furthermore, it is known that each GL-monoid is residuated, i.e., there exists a further binary operator $\mapsto: L \times L \to L$ (implication) on the lattice L, which satisfies the following condition:

$$\beta * \alpha \leq \gamma \iff \alpha \leq (\beta \mapsto \gamma) \text{ for any } \alpha, \beta, \gamma \in L.$$

Explicetly the implication is given by $\alpha \mapsto \beta = \bigvee\{\lambda \in L \mid \alpha * \lambda \leq \beta\}$.

As an example of GL-monoids one can mention Heyting algebras, i.e., any infinitely distributive lattice $(L, \leq)$ with the operation $* = \wedge$ forms a Heyting algebra and is a GL-monoid. Another useful class of GL-monoids is formed through the usage of continuous t-norms. It is known [2] that $([0,1], \leq, T)$ is a GL-monoid, where T is a t-norm on $[0,1]$ if and only if T is a continuous t-norm, i.e., it is continuous as a function in the usual interval topology $[0,1] \times [0,1]$.

In the following we denote by L both the GL-monoid, i.e. $L = (L, \leq, *)$, and the lattice L itself.

2.2 Fuzzy Categories

Definition 2 ([11]). *An L-fuzzy category is a tuple $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ where $(\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ is a crisp category and $\omega\colon \mathcal{O}b(\mathcal{C}) \to L$, $\mu\colon \mathcal{M}(\mathcal{C}) \to L$. Additionally ω and μ satisfy the following conditions:*

1. *if $f\colon X \to Y$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$;*
2. *$\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever $g \circ f$ is defined;*
3. *if $\mathrm{id}_X\colon X \to X$ is the identity morphism, then $\mu(\mathrm{id}_X) = \omega(X)$.*

Just as in classical category theory, we will write $f\colon X \to Y$ instead of $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$. We will refer to the object X as the domain and Y as the codomain of f. Furthermore, let us assume that L is known. In other words, instead of calling $\mathcal{C}$ an L-fuzzy category, we will refer to it as a fuzzy category.

We define $\mathcal{O}b_\alpha(\mathcal{C}) = \{X \in \mathcal{O}b(\mathcal{C}) \mid \omega(X) \geq \alpha\}$ and in a similar manner $\mathcal{M}_\alpha(\mathcal{C}) = \{f \in \mathcal{M}(\mathcal{C})\ \mu(f) \geq \alpha\}$. Objects from $\mathcal{O}b_\alpha(\mathcal{C})$ are called *α-objects*, and morphisms from $\mathcal{M}_\alpha(\mathcal{C})$ are called *α-morphisms*.

Now, consider a fuzzy category $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ and an object $X \in \mathcal{O}b(\mathcal{C})$. Intuitively, the value of $\omega(X)$ indicates the degree to which the potential object X truly belongs to the fuzzy category $\mathcal{C}$. Similarly, for a morphism $f \in \mathcal{M}(\mathcal{C})$, the value of $\mu(f)$ indicates the degree to which it is a morphism in the fuzzy category $\mathcal{C}$. Thus, we will use the terms "object degree" and "morphism degree".

Let us assume that $C = (\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ is a classical category. In this case, we can view the category $\mathcal{C}$ as a fuzzy category $(\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$, where ω and μ are defined as $\omega \equiv 1$ and $\mu \equiv 1$. Thus, classical categories and fuzzy categories that satisfy both of these additional conditions, i.e., $\omega \equiv 1$ and $\mu \equiv 1$, are essentially the same.

Furthermore, we can also obtain classical categories from fuzzy categories. If a fuzzy category $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ is given and $\iota \in L$ is an idempotent element, then it is possible to create a classical category $C_\iota = (\mathcal{O}b_\iota(\mathcal{C}), \mathcal{M}_\iota(\mathcal{C}), \circ)$, whose objects are all ι-objects and morphisms are ι-morphisms. We will call the categories created in this way threshold categories. There are two special cases for threshold categories. If $\iota = \top$, then $C_\top$ is called the *upper frame* of the fuzzy category $\mathcal{C}$. This threshold category will contain only those objects and morphisms whose degree of membership is $\top$ or $1_L = 1$. Of course, the case where the upper frame could coincide with the empty category is not excluded. The other interesting case is when $\iota = \bot$. In this case, the threshold category $\mathcal{C}_\bot$ is called the *bottom frame*. This threshold category will contain all objects whose degree of membership is at least $\bot$ or $0_L = 0$. So essentially it will contain all objects, but their degree of membership will be forgotten. Bottom frames are important because they form the basis for creating fuzzy categories from classical categories, as we will see later. Moreover, we can interpret the fuzzy category $\mathcal{C}$ as a kind of lattice, with the upper frame at the top and the bottom frame at the bottom. It will often happen that the upper frame and the bottom frame will be some well-known classical categories. In between, there could be classical categories of interesting character that have not been considered yet.

These threshold categories are in a sense classical subcategories to the fuzzy category, but it is also possible to create fuzzy subcategories.

Definition 3 ([11]). *Suppose that* $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ *is an L-fuzzy category. We call the L-fuzzy category* $\mathcal{C}' = (\mathcal{O}b(\mathcal{C}), \omega', \mathcal{M}(\mathcal{C}), \mu', \circ)$*, where* $\omega' \leq \omega$ *and* $\mu' \leq \mu$*, an L-fuzzy subcategory of* $\mathcal{C}$ *.*

A fuzzy category and its subcategory have the same class of objects and class of morphisms. The only difference is that the membership degrees of objects and morphisms in the subcategory are potentially lower.

In classical category theory, an important principle is duality, i.e., every definition is available in two versions. If we change the directions of morphisms in a classical category, i.e., arrows are reversed and the composition law is also reversed, then we can obtain the dual or opposite category. This construction is also possible in fuzzy categories, i.e., if $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ is a fuzzy category, then its dual fuzzy category will be $\mathcal{C}^{\text{op}} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ^{\text{op}})$, where $\circ^{\text{op}}$ is defined as $f \circ^{\text{op}} g = g \circ f$. In this paper, dual definitions will not be explored, but anyone familiar with classical category theory can develop them.

2.3 Functors

Definition 4 ([11]). *Let us assume that* $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega_{\mathcal{C}}, \mathcal{M}(\mathcal{D}), \mu_{\mathcal{C}}, \circ)$ *and* $\mathcal{D} = (\mathcal{O}b(\mathcal{D}), \omega_{\mathcal{D}}, \mathcal{M}(\mathcal{D}), \mu_{\mathcal{D}}, \circ)$ *are fuzzy categories and let* $F_1 \colon \mathcal{O}b(\mathcal{C}) \to \mathcal{O}b(\mathcal{D})$ *and* $F_2 \colon \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{D})$ *be maps. The quadruple* $F := (\mathcal{C}, \mathcal{D}, F_1, F_2)$ *is called a* δ*-functor* ($\delta \in L$) *if the following requirements are satisfied:*

1. *if* $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$*, then* $F_2(f) \in \mathcal{M}_{\mathcal{D}}(F_1(X), F_1(Y))$*;*
2. F_2 *preserves composition, i.e.,* $F_2(g \circ f) = F_2(g) \circ F_2(f)$*, if the composition* $f \circ g$ *is defined;*
3. F *preserves identities, i.e.,* $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$ *for any* $X \in \mathcal{O}b(\mathcal{C})$*;*
4. $\mu_{\mathcal{C}}(f) * \delta \leq \mu_{\mathcal{D}}(F_2(f))$ *for any* $f \in \mathcal{M}(\mathcal{C})$*.*

Of course, if $\delta' \leq \delta$, then a δ-functor is also a δ'-functor. In the special case of a 1-functor, it is a δ-functor for any $\delta \in L$.

Instead of writing $F_1(X)$ and $F_2(f)$, we will write $F(X)$ and $F(f)$ respectively. To describe a functor from $\mathcal{C}$ to $\mathcal{D}$, we will write $F \colon \mathcal{C} \to \mathcal{D}$. Similarly, we will write ω instead of $\omega_{\mathcal{C}}$ or $\omega_{\mathcal{D}}$, even if both of these mappings appear simultaneously in an equation, as the context will always make it clear in which category the membership degree is being measured. We will also treat μ and $\mu_{\mathcal{C}}$ or $\mu_{\mathcal{D}}$ in the same way.

The definition of a functor does not include conditions on the degrees of objects, but only on morphisms. One just has to recall that by definition, each object has an identity morphism.

Proposition 1. *If* $F \colon \mathcal{C} \to \mathcal{D}$ *is a* δ*-functor, then* $\omega_{\mathcal{C}}(X) * \delta \leq \omega_{\mathcal{D}}(F(X))$ *for all* $X \in \mathcal{O}b(\mathcal{C})$*.*

Proof. Since functors preserve identities and the degree of membership of an object coincides with the membership degree of the identity morphism, we have $\omega_{\mathcal{C}}(X) * \delta = \mu_{\mathcal{C}}(\mathrm{id}_X) * \delta \leq \mu_{\mathcal{D}}(F(\mathrm{id}_X)) = \mu_{\mathcal{D}}(\mathrm{id}_{F(X)}) = \omega_{\mathcal{D}}(F(X))$. □

Moreover, if the lattice L is a Heyting algebra and μ and ω are synced, i.e, for any $f \in \mathcal{M}_{\mathcal{C}}(X,Y)$, $\mu(f) = \omega(X) \wedge \omega(Y)$, then $\mu(f) \geq \alpha$ if and only if $\omega(X) \geq \alpha$ and $\omega(Y) \geq \alpha$.

Proposition 2. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a δ-functor and δ is an idempotent element, then the restriction of the functor $F_\delta\colon \mathcal{C}_\delta \to \mathcal{D}_\delta$ to the threshold categories $\mathcal{C}_\delta$ and $\mathcal{D}_\delta$ is a classical functor between these categories.*

Proof. To prove this, it suffices to show that $F_\delta(\mathcal{C}_\delta) \subseteq \mathcal{D}_\delta$, since then we cannot have a situation where the codomain of one of the δ-morphisms does not belong to the threshold category. Suppose $f \in \mathcal{M}_\delta(\mathcal{C})$. Then $\mu_{\mathcal{D}}(F(f)) \geq \mu_{\mathcal{C}}(f) * \delta \geq \delta * \delta = \delta$. From Proposition 1 we can infer that for any $X \in \mathcal{C}_\delta$ there will also be $\omega_{\mathcal{D}}(F(X)) \geq \delta$. □

The converse is not always true except in the trivial case, i.e., if $F\colon \mathcal{C}_\perp \to \mathcal{D}_\perp$ is a classical functor between bottom frame categories, then F is a 0-functor from $\mathcal{C}$ to $\mathcal{D}$. Combining this with Proposition 2 we can describe 0-functors.

Proposition 3. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a 0-functor if and only if the restriction of the functor F to the bottom frame categories $F_\perp\colon \mathcal{C}_\perp \to \mathcal{D}_\perp$ is a classical functor between the corresponding bottom frame categories.*

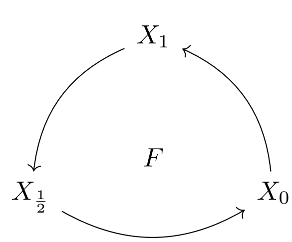
We can combine functors in fuzzy categories to form new functors.

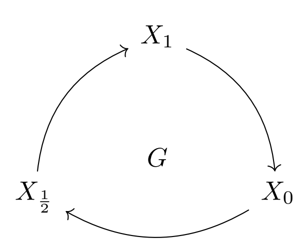
Proposition 4. *If $F\colon \mathcal{A} \to \mathcal{B}$ is a λ-functor and $G\colon \mathcal{B} \to \mathcal{C}$ is a δ-functor, then their composition $GF\colon \mathcal{A} \to \mathcal{C}$ is a $\lambda * \delta$-functor.*

Proof. Suppose $f \in \mathcal{M}(\mathcal{A})$. Since F is a λ-functor and G is a δ-functor, then $\mu_{\mathcal{C}}(GF(f)) \geq \lambda * \mu_{\mathcal{B}}(G(f)) \geq (\lambda * \delta) * \mu_{\mathcal{A}}(f)$. □

Of course, this does not mean that GF cannot have a higher degree.

Example 1. Suppose we consider a fuzzy discrete category $\mathcal{C}$ with three objects $X_1, X_{\frac{1}{2}}, X_0$ whose degrees coincide with their indices. We can define functors $F, G : \mathcal{C} \to \mathcal{C}$ by the corresponding diagrams:





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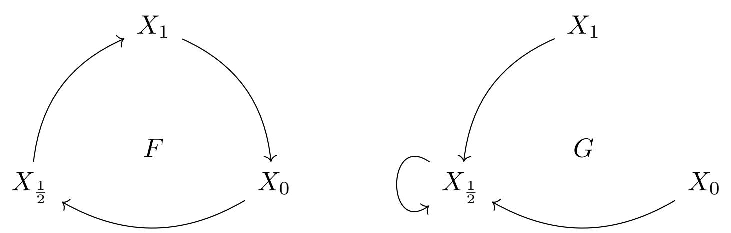
As soon as a functor represents an object whose membership degree is not 0 by an object whose membership degree is 0, then that functor is a 0-functor, so we can conclude that F and G are 0-functors. However, by composing these functors into FG and GF we obtain 1-functors as identity functors. From this example we can see that if the degree of the functors FG and GF is known, it is not possible to infer any information about the functors F and G with respect to their degrees unless additional information is given.

The question of inferring the degree of a functor is quite important. In classical category theory there are several theorems which deduce the existence of a particular (left adjoint) functor. Although these theorems can be proven in fuzzy category theory, it will not guarantee the degree of the functor, so we end up with only a 0-functor. It would be desirable to find sufficient and necessary conditions that would guarantee the degree of a functor if some of its compositions and other conditions are known. In classical category theory all functors are considered 1-functors in the fuzzy sense, so no such problems arise there.

Moreover, even if the degrees of the functors G, FG and GF are known, it is not possible to conclude anything about the functor F without additional information.

Example 2. Suppose that the fuzzy category $\mathcal{C}$ from Example 1 is given. Let us define the functors $F, G\colon \mathcal{C} \to \mathcal{C}$ with the following diagrams:



We can notice that G is a $\frac{1}{2}$-functor, FG is a 1-functor, FG is a $\frac{1}{2}$-functor, but F is a 0-functor.

There is a simple sufficient condition, but of course it is not necessary, to determine the degree of the functor F if some information about G and GF is known.

Definition 5. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors between fuzzy categories. We call a functor G non-increasing with respect to F if $\mu_{\mathcal{B}}(F(f)) \geq \mu_{\mathcal{C}}(GF(f))$ for all $f \in \mathcal{M}(\mathcal{A})$.*

Although this is a strong condition, in a way it will be natural when we apply it to initial objects in comma categories.

Proposition 5. *Suppose $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are functors between fuzzy categories, and GF is a δ-functor. If G is non-increasing with respect to F, then F is a δ functor.*

Proof. Since G is non-increasing with respect to F, for any morphism $f \in \mathcal{M}(\mathcal{A})$ we have $\mu_{\mathcal{B}}(F(f)) \geq \mu_{\mathcal{C}}(GF(f)) \geq \delta * \mu_{\mathcal{A}}(f)$. □

3 Adjoint functors

From now on we assume that the lattice L is a linearly ordered Heyting algebra. Furthermore, we assume that we are dealing with fuzzy categories in which the membership degrees of morphisms are synced with the membership degree of objects, i.e., $\mu(f) = \omega(X) \wedge \omega(Y)$ for all $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$.

3.1 Natural transformations

Definition 6. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be δ-functors. We call a family of morphisms $\eta\colon F \Rightarrow G$, where for each $X \in \mathcal{O}b(\mathcal{C})$ the component $\eta_X\colon F(X) \to G(X)$ is a morphism between objects of $\mathcal{D}$, a λ-natural transformation from the functor F to the functor G if*

1. $\omega_{\mathcal{C}}(X) * \lambda \leq \mu_D(\eta_X)$ *for any* $X \in \mathcal{O}b(\mathcal{C})$ *;*
2. $\eta_Y \circ F(f) = G(f) \circ \eta_X$ *for any* $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$.

One could try to define in a straightforward way that η is a λ-natural transformation if $\mu(\eta_X) \geq \lambda$ for all $x \in X$, but in this case we run into the problem that if a category contains a 0-object Y then $\mu(\eta_Y) = 0$, since the degrees of morphisms are bounded by its domain and codomain. This means that in such a definition almost all natural transformations are 0-natural, unless 0-objects do not exist in the fuzzy category.

There is one additional reason for giving natural transformations a condition similar to the δ-functor condition on the degrees of membership of morphisms. In classical category theory, natural transformations between functors are respectively morphisms between functors in the category of functors. If we define the degree of a δ-functor in this category to be δ, then we expect that the degree of a natural transformation (morphism between functors) would not exceed the minimum of the degrees of membership of the domain and codomain (functors). This means that in the category of functors we expect $\mu(\eta) \leq \omega(F) \wedge \omega(G)$, where η is a natural transformation from functor F to functor G. If the degree of a functor in this category coincides with the largest δ and δ', for which F is a δ-functor and G is a δ'-functor, then $\mu(\eta) \leq \delta \wedge \delta'$. Moreover, since we are dealing here with fuzzy categories for which the degree of membership of morphisms is synced to the objects, we can unambiguously characterise the degree of naturalness of η.

Proposition 6. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a δ_1-functor, $G\colon \mathcal{C} \to \mathcal{D}$ is a δ_2-functor, $\eta\colon F \Rightarrow G$ is a λ-natural transformation from F to G, then $\lambda \geq \delta_1 \wedge \delta_2$.*

Proof. Since $\eta_X \in \mathcal{M}_{\mathcal{D}}(F(X), G(X))$, then by Proposition 1

$$\mu(\eta_X) = \omega(F(X)) \wedge \omega(G(X)) \geq (\delta_1 \wedge \omega(X)) \wedge (\delta_2 \wedge \omega(X)) = (\delta_1 \wedge \delta_2) \wedge \omega(X).$$

So η is at least a $(\delta_1 \wedge \delta_2)$-natural transformation. □

This implies that we would have $\lambda = \delta_1 \wedge \delta_2$ in the category of functors.

Now through the use of the unit and co-unit we can define adjoint functor pairs naturally in fuzzy categories.

Definition 7. *Suppose $G\colon \mathcal{C} \to \mathcal{D}$ is a δ-functor and $F\colon \mathcal{D} \to \mathcal{C}$ is a δ-functor. The functors F and G form an α-adjoint functor pair (written as $F \dashv_\alpha G$) and F is called the left adjoint of the functor G (respectively G is the right adjoint of the functor F) if*

1. *there exist α-natural transformations $\eta\colon \mathrm{id}_\mathcal{D} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow \mathrm{id}_\mathcal{C}$;*
2. *transformations η and ε satisfy the triangle equations, i.e. $\mathrm{id}_F = \varepsilon F \circ F\eta$ and $\mathrm{id}_G = G\varepsilon \circ \eta G$.*

Classical category theory contains a plentiful amount of theorems that guarantee the existence of adjoint functor pairs. Although our goal is to prove the existence of a functor (left adjoint), it would be preferable its membership degree would be guaranteed to some extent. Otherwise all resulting functors could be just 0-functors and we would lose information about the degree of objects and morphisms by transforming them with these functors.

3.2 Sufficient condition for an adjoint functor

While the notion of non-increasing was previously applied to functors, we can now do so to objects from comma categories which are morphisms. The membership degrees of the objects in comma categories naturally are inherited from the target category, i.e., if $S\colon \mathcal{A} \to \mathcal{C}$ and $T\colon \mathcal{B} \to \mathcal{C}$ are functors and $S \downarrow T$ is a comma category, then then the membership degree for all objects (morphisms) (A, B, h) of the comma category coincides with the degree of the morphism h from the category $\mathcal{C}$.

Definition 8. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are fuzzy categories, $F\colon \mathcal{C} \to \mathcal{B}$ is a functor and B is an object from the category $\mathcal{B}$. We call an object (B, A, h) from the comma category $B \downarrow F$ non-increasing if $\omega(A) \geq \omega(F(A))$*

Theorem 1. *If $G\colon \mathcal{A} \to \mathcal{B}$ is a δ-functor and for any object $B \in \mathcal{B}$ in the comma category $B \downarrow G$ there exists a non-increasing initial object (B, A, h) for which $\mu(h) \geq B \wedge \delta$, then there exists a δ-functor $F\colon \mathcal{B} \to \mathcal{A}$ such that $F \dashv_\delta G$.*

Proof. The existence of the left adjoint F and the natural transformations η, ε is a well known fact (see, e.g., [4]). All that remains is to show that the extra conditions imposed in the formulation guarantee the degree of the functor and adjoint pair.

First, let us note that $\eta\colon \mathrm{id}_\mathcal{B} \Rightarrow GF$ is a δ-natural transformation. This follows because for each $B \in \mathcal{C}(\mathcal{B})$ we have $\omega(B, F(B), \eta_B) = \mu(\eta_B) \geq \delta \wedge \omega(B)$ under the conditions of the theorem on the initial object. From this we can imply that GF is a δ-functor since

$$\begin{aligned}\mu(GF(f)) \geq \mu(GF(f)) \wedge \mu(f) &= (\omega(GF(B)) \wedge \omega(GF(B'))) \wedge (\mu(B) \wedge \mu(B')) \\ &= (\omega(GF(B)) \wedge \omega(B)) \wedge (\omega(GF(B')) \wedge \omega(B')) \\ &= \mu(\eta_B) \wedge \mu(\eta_{B'}) \\ &\geq (\delta \wedge \omega(B)) \wedge (\delta \wedge \omega(B')) = \delta \wedge \mu(f).\end{aligned}$$

Now let us reason that F is a δ-functor. Suppose that $f \in \mathcal{M}_{\mathcal{B}}(B, B')$ is an arbitrary morphism. Since η_B and η'_B are given to be non-increasing initial objects of the comma category $B \downarrow G$, we know that $\mu(F(f)) = \omega(F(B)) \wedge \omega(F(B')) \geq \omega(GF(B)) \wedge \omega(GF(B')) = \mu(GF(f))$. Combining the with the fact that GF is a δ-functor, we get that $\mu(F(f)) \geq \mu(GF(f)) \geq \delta \wedge \mu(f)$, i.e., F is a δ-functor.

Now it remains to show that $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{A}}$ is a δ-natural transformation. Since F and G are δ-functors, then the composition FG is also a δ-functor. From this we can infer that for each $A \in \mathcal{C}(\mathcal{A})$ we have $\mu(\varepsilon_A) = \omega(A) \wedge \omega(FG(A)) \geq \delta \wedge \omega(A)$.

We conclude that F is a δ-functor such that $F \dashv_\delta G$. □

Similar to the way we can obtain a crisp functor from δ-functors for each idempotent $\iota \leq \delta$, we can obtain in such a way a class of crisp adjoint functor pairs from a single pair of fuzzy adjoint functors.

Theorem 2. *If $F \dashv_\lambda G$ is a λ-adjoint functor pair between the fuzzy categories $\mathcal{C}$ and $\mathcal{D}$, and $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$ are λ-functors, then the restriction of these functors to the threshold categories $F_\delta\colon \mathcal{C}_\lambda \to \mathcal{D}_\lambda$ and $F_\lambda\colon \mathcal{C}_\lambda \to \mathcal{D}_\lambda$ form a crisp adjoint functor pair $F_\lambda \dashv G_\lambda$ between the corresponding threshold categories.*

Proof. If $F \dashv_\lambda G$ is a λ-adjoint functor pair between categories $\mathcal{C}$ and $\mathcal{D}$, then by Proposition 2 F_λ and G_λ are functors between their respective threshold categories $\mathcal{C}_\lambda$ and $\mathcal{D}_\lambda$.

Now we just need to show the natural transformations that will form the adjoint pair. Let η and ε be the λ-natural transformations, which are induced by the λ-adjoint pair. If we consider the restriction of η to $\eta_\lambda\colon \mathrm{id}_{\mathcal{D}_\lambda} \to (GF)_\lambda$ to the functors $\mathrm{id}_{\mathcal{D}_\lambda}$ and $(GF)_\lambda$, it suffices to show that for each component η_λ its codomain is contained in the threshold category $\mathcal{D}_\lambda$. Thus we would be able to infer that η_λ is a crisp natural transformation.

Suppose that $f \in \mathcal{M}_{\mathcal{D}_\lambda}(X, X')$. Since $\mu(f) = \omega(X) \wedge \omega(X')$, then $\omega(X) \geq \lambda$ or $X \in \mathcal{D}_\lambda$. In such case $\omega(GF(X)) \geq \lambda \wedge \omega(X) \geq \lambda \wedge \lambda = \lambda$, as in $GF(X) \in \mathcal{D}_\lambda$. Which means that all of the natural transformation diagrams still commute.

$$\begin{array}{ccc} X & \xrightarrow{(\eta_\lambda)_X} & (GF)_\lambda(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (GF)_\lambda(f)} \\ X' & \xrightarrow[(\eta_\lambda)_{X'}]{} & (GF)_\lambda(X') \end{array}$$

With similar arguments about the domain, one can show that $\varepsilon_\lambda \colon (FG)_\lambda \Rightarrow \mathrm{id}_{\mathcal{C}_\lambda}$ is a natural transformation. Furthermore, η_λ and ε_λ satisfy the triangle equalities, because η and ε satisfy them and the degree of morphisms does not decrease under composition (axiom of fuzzy categories).

We conclude that $F_\lambda \dashv G_\lambda$. □

4 Conclusion

In this paper we have taken a step to further develop fuzzy category theory. The current tools of crisp category theory cannot directly describe situations where some local context is known of a model or even where we want to expand the class of morphisms or objects by easing the restrictions on them whilst also keeping track to what degree a potential object or morphism is a real object or morphism. Although every single result that is true in crisp category theory is also true in fuzzy category theory, the methods and transformations might not respect the membership degree of objects and morphisms. Therefore new tools have to be developed that guarantee (to a certain degree) that data will not be lost. Currently we have only provided a sufficient condition for a left adjoint functor that respects membership degrees up to a certain point. It still is an open question if there is a necessary condition for this existence outside of the trivial cases (1-functors and 0-functors). Furthermore, it remains to be seen how this result can be further developed for locally small, complete categories using weak initial objects. This would require a sensible definition of some sort of limits and continuity which also respects membership degrees.

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