Bipolar inequations on complete distributive symmetric residuated lattices ⋆

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Abstract. The resolution of bipolar equations with usual max-triangular norm compositions (Gödel, product and Łukasiewicz) and the standard negation have been extensively studied. Recent theoretical advances extend the algebraic framework of bipolar equations to the context of non-linear lattices, considering at the same time a general class of triangular norms, an arbitrary involutive negation and a join-irreducible element of the lattice as independent term. This paper studies the resolution of bipolar inequations within the latter algebraic setting, which leads to an alternative strategy to compute the solutions set of bipolar equations.

Keywords: Bipolar inequation · Residuated lattice · Involutive negation · Irreducible element

1 Introduction

Bipolar fuzzy relation equations (BFRE), introduced by Freson et al. [14], have been studied in some particular cases in the literature, including BFRE defined with the standard negation and the max-min composition [14, 17, 18], the max-product composition [3, 7] and the max-Łukasiewicz composition [19, 23, 24]. Additionally, the particular case of BFRE defined with the product negation and the max-product composition has also been studied in [5, 6].

Recently, a general kind of BFRE has been considered in [8], assuming a non-linear carrier, an involutive negation and a sup-$*$ composition being $*$ an operator with residuum. To be precise, the underlying algebraic structure is a complete distributive symmetric residuated lattice, i.e. complete distributive residuated lattice endowed with an involutive negation. The key point of the solving strategy followed in [8] is that the right-hand side of the BFRE is a join-irreducible element of the underlying lattice.

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In this paper, we address the resolution of bipolar sup-$*$ inequations defined in complete distributive symmetric residuated lattices with join-irreducible right-hand side. This study yields to a strategy to compute the complete solution set of a bipolar sup-$*$ equation, which is different from the one given in [8]. Since BFRE are actually based on bipolar sup-$*$ equations, this contribution will be useful to take the first steps towards the solvability of BFRE with a join-reducible right-hand side.

2 Preliminaries

This section includes some basic notions about lattice theory [1, 9, 15], which play a key role in the resolution of a bipolar equation and the description of its solution set [8]. To begin with, the definitions of complete lattice and distributive lattice are shown.

Definition 1. *Let $(L, \preceq)$ be a lattice. We say that:*

- $(L, \preceq)$ *is a* complete lattice *if* $\bigvee S \in L$ *and* $\bigwedge S \in L$*, for all* $S \subseteq L$*.*
- $(L, \preceq)$ *is a* distributive lattice *if, for all* $x, y, z \in L$*:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Note that, any complete lattice has a top and a bottom element, which will be denoted as $\top$ and $\bot$, respectively.

The solvability of bipolar equations in a complete distributive symmetric residuated lattice was studied in [8], where two additional requirements are demanded: the condition of join-irreducible element for the right-hand side of the equation and the condition of homomorphism of the underlying lattice for the operator $*$. These concepts are formally defined below.

Definition 2. *Let $(L, \preceq)$ be a complete lattice.*

- *An element* $x \in L$ *is* join-irreducible *if* $x \neq \bot$ *and* $x = a \vee b$ *implies* $x = a$ *or* $x = b$*, for all* $a, b \in L$*.*
- *A* supremum (infimum) homomorphism *is a mapping* $f \colon L \to L$ *which preserves the supremum (infimum) of any nonempty subset, i.e.* $f(\bigvee A) = \bigvee f(A)$ $(f(\bigwedge A) = \bigwedge f(A))$*, for all* $A \in \mathcal{P}(L) \setminus \{\varnothing\}$*, where* $\mathcal{P}(L)$ *is the powerset of* L*.*
- *A* lattice homomorphism*, or simply* homomorphism*, is a supremum and infimum homomorphism.*
- *An* involutive negation *is an order-reversing mapping* $\neg \colon L \to L$ *such that* $\neg\neg x = x$ *for all* $x \in L$*.*

Observe that, the concept of lattice homomorphism in Definition 2, taken from [11], does not coincide with the standard definition [1, 9], which only required to be join-preserving and meet-preserving for pairs of elements of L.

Additionally, the usual definition of negation operator refers to any order-reversing mapping $\neg \colon L \to L$ satisfying the boundary conditions $\neg\top = \bot$ and

$\neg\bot = \top$. Indeed, such conditions hold for any involutive negation, so they are properly omitted in Definition 2.

Considering a purely algebraic viewpoint, residuated lattices were presented as a lattice endowed with a residuated pair [12]. Residuated pairs consist of two binary operators capable of generalizing the philosophy of the Modus Ponens inference rule [16].

Definition 3. *A* residuated lattice *is a tuple* $(L, \preceq, *, \rightarrow)$ *where* $(L, \preceq)$ *is a lattice with top element* $\top$*,* $(L, *, \top)$ *is a commutative monoid and* $(*, \rightarrow)$ *is a* residuated pair*, that is, for all* $x, y, z \in L$*:*

$$x * y \preceq z \qquad \text{if and only if} \qquad x \preceq y \rightarrow z \tag{1}$$

Equivalence (1) *is called residuated property.*

Although symmetric residuated lattices were firstly presented in [2], they have newly been characterized in [8]. In this paper, we have chosen to define this concept in terms of such characterization, for the sake of simplicity.

Definition 4. *A tuple* $(L, \preceq, *, \rightarrow, \neg)$ *is a complete distributive symmetric residuated lattice (CDSRL) if* $(L, \preceq, *, \rightarrow)$ *is a complete distributive residuated lattice and* $\neg\colon L \rightarrow L$ *is an involutive negation.*

After recalling the previous lattice-theoretical notions, in order to make the paper self-contained, we will present the formal definition of bipolar equation. A detailed study about these equations and their solvability can be found in [8].

Definition 5. *Let* $(L, \preceq, *, \rightarrow, \neg)$ *be a CDSRL,* b *a join-irreducible element of* L *and* $a_j^+, a_j^- \in L$*, for each* $j \in \{1, \dots, m\}$*. A* bipolar equation with sup-$*$ composition*, or simply, a* bipolar sup-$*$ equation *is an expression of the form:*

$$\bigvee_{j\in\{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) = b \tag{2}$$

where $x_1, \dots, x_m \in L$ *are unknown values.*

Now, we will detail the notation employed to describe the solution set of Equation (2). First of all, we consider the mappings $\rightarrow$ and $\rightsquigarrow$ associated with $(L, \preceq, *)$ which are defined as follows:

$$a \rightarrow b = \max\{x \in L \mid a * x \preceq b\}$$
$$a \rightsquigarrow b = \inf\{x \in L \mid b \preceq a * x\}$$

These mappings were firstly introduced in [13]. It is convenient to mention that, applying Equivalence (1), we obtain that $\rightarrow$ is the residuated implication of $*$. In addition, the mapping $\rightsquigarrow$ plays a key role in the description of the solutions of a sup-$*$ equation [10], and as a consequence, it is also useful for the case of bipolar sup-$*$ equations.

Consider the bipolar sup-$*$ equation (2) and the natural extension of the negation $\neg$, supremum $\vee$ and infimum $\wedge$ operators, from L to L^m.

- The sets S^+ and S^- are defined as

$$S^+ = \{j \in \{1,\dots,m\} \mid b \preceq a_j^+, a_j^+ \rightsquigarrow b \preceq a_j^+ \to b\}$$
$$S^- = \{j \in \{1,\dots,m\} \mid b \preceq a_j^-, a_j^- \rightsquigarrow b \preceq a_j^- \to b\}$$

- Given $j \in \{1,\dots,m\}$, we define

$$s_j^+ = (\bot,\dots,\bot, a_j^+ \rightsquigarrow b, \bot,\dots,\bot)$$
$$s_j^- = (\bot,\dots,\bot, a_j^- \rightsquigarrow b, \bot,\dots,\bot)$$

  where $a_j^+ \rightsquigarrow b$, $a_j^- \rightsquigarrow b$ correspond to the j-th position of s_j^+, s_j^-, respectively.
- The tuples g^+ and g^- belonging to L^m are defined as

$$g^+ = (a_1^+ \to b,\dots,a_m^+ \to b)$$
$$g^- = (a_1^- \to b,\dots,a_m^- \to b)$$

Finally, before presenting the analytic expression of the solution set of a solvable bipolar sup-$*$ equation, we recall that the unary mappings $*_x, *_y \colon L \to L$ given by $*_x(y) = x * y$ and $*_y(x) = x * y$, for all $x, y \in L$, are called partial mappings of $*$.

Theorem 1 ([8]). *Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1,\dots,m\}$, and b a join-irreducible element of L. If the bipolar sup-$*$ equation* (2) *is solvable, then its solution set is equal to*

$$D = \Big(\bigcup_{j\in S^+} [s_j^+ \vee \neg g^-, g^+]\Big) \cup \Big(\bigcup_{j\in S^-} [\neg g^-, g^+ \wedge \neg s_j^-]\Big)$$

As shown next, Theorem 1 leads to a simple necessary condition for the solvability of a bipolar sup-$*$ equation.

Corollary 1 ([8]). *Let $(L, \preceq, *, \to, \neg)$ be a CDSRL such that the partial mappings of $*$ are homomorphisms. Let $a_j^+, a_j^- \in L$, for each $j \in \{1,\dots,m\}$, and b a join-irreducible element of L. If the bipolar sup-$*$ equation* (2) *is solvable, then the inequality $\neg g^- \preceq g^+$ holds.*

3 Bipolar inequations

The concept of bipolar inequation is formalized next. In order to extend the philosophy of the solving strategies developed in [8] and [10], we assume here a CDSRL as the underlying algebraic structure of bipolar inequations. Nevertheless, the mathematical requirements necessary to define this notion are weaker. Namely, it suffices to consider a join semilattice.

Definition 6. *Let* $(L, \preceq, *, \rightarrow, \neg)$ *be a CDSRL and* $a_j^+, a_j^-, b \in L$, *for each* $j \in \{1, \dots, m\}$. *A* bipolar inequation with sup-$*$ composition*, or simply, a* bipolar sup-$*$ inequation *is an expression of the form:*

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) \preceq b \tag{3}$$

or of the form:

$$b \preceq \bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) \tag{4}$$

where $x_1, \dots, x_m \in L$ *are unknown values.*

In this paper, we will present some results concerning the resolution of bipolar sup-$*$ inequations of the form (3) and of the form (4). As it will be shown, there is a marked difference between the form of the solution set of Inequation (3) and the form of the solution set of Inequation (4). Additionally, such difference is transferred to the necessary or sufficient conditions of the solvability of Inequations (3) and (4).

The next theorem shows the solution set of a bipolar sup-$*$ inequation of the form (3). The foundations of the theorem consists of splitting the expression

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) \preceq b$$

into two parts, resulting in

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \quad \vee \bigvee_{j \in \{1,\dots,m\}} (a_j^- * \neg x_j) \preceq b$$

As a result, the solution set of Inequation (3) can be written as the intersection of the solution set of

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^+ * x_j) \preceq b$$

and the solution set of

$$\bigvee_{j \in \{1,\dots,m\}} (a_j^- * \neg x_j) \preceq b$$

Theorem 2. *Let* $(L, \preceq, *, \rightarrow, \neg)$ *be a CDSRL and* b *a join-irreducible element of* L. *If the partial mappings of* $*$ *are supremum-morphisms, then the solution set of Inequation* (3) *is* $[\neg g^-, g^+]$.

Observe that, if Inequation (3) is not bipolar, i.e. $a_j = \perp$ for all $j \in \{1, \dots, m\}$, then Theorem 2 implies that its solution set is $[0, g^+]$, being $0 = (\perp, \dots, \perp)$. Clearly, this particular case coincides with Proposition 16 of [10].

An interesting consequence of Theorem 2 is that, the solution set of Inequation (3) is not empty if and only if $\neg g^- \preceq g^+$. In other words, the solvability of Inequation (2) is characterized by the satisfiability of such inequality.

Proposition 1. *Let* $(L, \preceq, *, \to, \neg)$ *be a CDSRL and* b *a join-irreducible element of* L*. If the partial mappings of* $*$ *are supremum-morphisms, Inequation* (3) *is solvable if and only if* $\neg g^- \preceq g^+$*.*

Naturally, the bipolar sup-$*$ equation (2) inherits the inequality $\neg g^- \preceq g^+$ as a necessary condition of its solvability. Indeed, this result coincides with Corollary 1.

In what regards the resolution of the bipolar sup-$*$ inequation (4), notice that

$$b \preceq \bigvee_{j\in\{1,\dots,m\}} (a_j^+ * x_j) \vee (a_j^- * \neg x_j)$$

if and only if $b \preceq (a_j^+ * x_j) \vee (a_j^- * \neg x_j)$ for some $j \in \{1, \dots, m\}$. Equivalently, $b \preceq (a_j^+ * x_j)$ or $b \preceq (a_j^- * \neg x_j)$ for some $j \in \{1, \dots, m\}$. Hence, the solution set of Inequation (4) coincides with the union of the solution set of all inequalities of the form $b \preceq (a_j^+ * x_j)$ and $b \preceq (a_j^- * \neg x_j)$, where $j \in \{1, \dots, m\}$.

For the sake of readability, consider fixed the sets

$$\begin{aligned} S^{\oplus} &= \{j \in \{1, \dots, m\} \mid b \preceq a_j^+\} \\ S^{\ominus} &= \{j \in \{1, \dots, m\} \mid b \preceq a_j^-\} \end{aligned}$$

Besides, given $j \in \{1, \dots, m\}$, we define

$$\begin{aligned} s_j^+ &= (\bot, \dots, \bot, a_j^+ \rightsquigarrow b, \bot, \dots, \bot) \\ s_j^- &= (\bot, \dots, \bot, a_j^- \rightsquigarrow b, \bot, \dots, \bot) \end{aligned}$$

where $a_j^+ \rightsquigarrow b$, $a_j^- \rightsquigarrow b$ correspond to the j-th position of s_j^+, s_j^-, respectively.

Theorem 3. *Let* $(L, \preceq, *, \to, \neg)$ *be a CDSLR and* b *a join-irreducible element of* L*. If the partial mappings of* $*$ *are infimum-morphisms, the solution set of Inequation* (4) *is:*

$$\left(\bigcup_{j\in S^{\oplus}} [s_j^+, 1]\right) \cup \left(\bigcup_{j\in S^{\ominus}} [0, \neg s_j^-]\right)$$

Although the analytic form of the solution set of Inequation (4) is considerably more complex than the solution set of Inequation (3), the existence of solutions of Inequation (4) can be characterized in a simple manner in terms of its coefficients.

Proposition 2. *Let* $(L, \preceq, *, \to, \neg)$ *be a CDSRL and* b *a join-irreducible element of* L*. If the partial mappings of* $*$ *are infimum-morphisms, Inequation* (4) *is solvable if and only if there exists* $j \in \{1, \dots, m\}$ *such that* $b \preceq a_j^+$ *or* $b \preceq a_j^-$*.*

Clearly, due to the reflexivity and the antisymmetry of the partial order $\preceq$, the next statement holds:

$$a = b \quad \text{if and only if} \quad a \preceq b \quad \text{and} \quad b \preceq a$$

Hence, the solution set of Equation (2) can be written as the intersection of the solution sets of Inequations (3) and (4). It is important to highlight that this strategy was already used to solve sup-$*$ equations in complete distributive residuated lattices [10].

According to Theorems 2 and 3, the solution set of Equation (2) equals:

$$
\begin{aligned}
&[\neg g^-, g^+] \cap \left(\left(\bigcup_{j \in S^\oplus} [s_j^+, 1] \right) \cup \left(\bigcup_{j \in S^\ominus} [0, \neg s_j^-] \right) \right) \\
&\overset{(1)}{=} \left([\neg g^-, g^+] \cap \bigcup_{j \in S^\oplus} [s_j^+, 1] \right) \cup \left([\neg g^-, g^+] \cap \bigcup_{j \in S^\ominus} [0, \neg s_j^-] \right) \\
&\overset{(2)}{=} \left(\bigcup_{j \in S^\oplus} [\neg g^-, g^+] \cap [s_j^+, 1] \right) \cup \left(\bigcup_{j \in S^\ominus} [\neg g^-, g^+] \cap [0, \neg s_j^-] \right) \\
&\overset{(3)}{=} \left(\bigcup_{j \in S^\oplus} [\neg g^- \vee s_j^+, g^+ \wedge 1] \right) \cup \left(\bigcup_{j \in S^\ominus} [\neg g^- \vee 0, g^+ \wedge \neg s_j^-] \right) \\
&\overset{(4)}{=} \left(\bigcup_{j \in S^\oplus} [\neg g^- \vee s_j^+, g^+] \right) \cup \left(\bigcup_{j \in S^\ominus} [\neg g^-, g^+ \wedge \neg s_j^-] \right)
\end{aligned}
$$

where (1) is obtained by the distributivity of $(L, \preceq)$, (2) by De Morgan's laws, (3) by the intersection of intervals and (4) by the inequalities $g^+ \preceq 1$ and $0 \preceq \neg g^-$.

Note that, the latter expression in the chain of equalities coincides with the solution set given in Theorem 1, except for the expression of the index sets $S^\oplus$, $S^\ominus$ and S^+, S^-, respectively. However, it can be is easily seen that given $j \in S^\oplus \setminus S^+$, the interval $[\neg g^- \vee s_j^+, g^+]$ is empty. Equivalently, the interval $[\neg g^-, g^+ \wedge \neg s_j^-]$ is empty for each $j \in S^\ominus \setminus S^-$. Hence, both expressions are actually equivalent.

Next example illustrates how to compute the solution set of bipolar equations by using the approach introduced in this paper.

Example 1. Let $([0,1], \leq, \wedge, \rightarrow_\wedge, \neg_S)$ be a CDSRL composed of the Gödel residuated pair and the standard negation, which are defined as:

$$
a \wedge b = \min\{a, b\} \qquad a \rightarrow_\wedge b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases} \qquad \neg_S a = 1 - a
$$

for all $a, b \in [0,1]$, and $\rightsquigarrow_\wedge \colon [0,1] \times [0,1] \rightarrow [0,1]$ be a mapping defined as follows:

$$
a \rightsquigarrow_\wedge b = \inf\{x \in [0,1] \mid b \leq a \wedge x\} = \begin{cases} b & \text{if } b \leq a \\ 1 & \text{otherwise} \end{cases}
$$

for all $a, b \in [0,1]$. Consider the bipolar equation with max-min composition given below:

$$(0.4 \wedge x_1) \vee (0.5 \wedge \neg_S x_1) \vee (0.6 \wedge x_2) \vee (0.4 \wedge \neg_S x_2) \vee (0.7 \wedge x_3) \vee (0.8 \wedge \neg_S x_3) = 0.6 \quad \text{(A)}$$

Next bipolar inequations need to be solved in order to obtain the solution set of Equation (A):

$$(0.4 \wedge x_1) \vee (0.5 \wedge \neg_S x_1) \vee (0.6 \wedge x_2) \vee (0.4 \wedge \neg_S x_2) \vee (0.7 \wedge x_3) \vee (0.8 \wedge \neg_S x_3) \leq 0.6$$

$$0.6 \leq (0.4 \wedge x_1) \vee (0.5 \wedge \neg_S x_1) \vee (0.6 \wedge x_2) \vee (0.4 \wedge \neg_S x_2) \vee (0.7 \wedge x_3) \vee (0.8 \wedge \neg_S x_3)$$

From now on, the previous bipolar inequations will be labelled as Inequations (A1) and (A2), respectively. It is easy to see that the hypothesis required in Theorems 2 and 3 are satisfied, and as a consequence:

- The solution set of Inequation (A1) is $[\neg_S g^-, g^+] = [(0,0,0.4),(1,1,0.6)]$.

$$\begin{aligned} g^+ &= (0.4 \to_\wedge 0.6, 0.6 \to_\wedge 0.6, 0.7 \to_\wedge 0.6) = (1,1,0.6) \\ g^- &= (0.5 \to_\wedge 0.6, 0.4 \to_\wedge 0.6, 0.8 \to_\wedge 0.6) = (1,1,0.6) \\ \neg_S g^- &= (\neg_S 1, \neg_S 1, \neg_S 0.6) = (0,0,0.4) \end{aligned}$$

- The solution set of Inequation (A2) is $\left(\bigcup_{j \in S^\oplus} [s_j^+, 1]\right) \cup \left(\bigcup_{j \in S^\ominus} [0, \neg s_j^-]\right)$. Considering the next index sets

$$\begin{aligned} S^\oplus &= \{j \in \{1,2,3\} \mid 0.6 \preceq a_j^+\} = \{2,3\} \\ S^\ominus &= \{j \in \{1,2,3\} \mid 0.6 \preceq a_j^-\} = \{3\} \end{aligned}$$

and the following tuples

$$\begin{aligned} s_2^+ &= (0, a_2^+ \rightsquigarrow_\wedge b, 0) = (0, 0.6 \rightsquigarrow_\wedge 0.6, 0) = (0, 0.6, 0) \\ s_3^+ &= (0, 0, a_3^+ \rightsquigarrow_\wedge b) = (0, 0, 0.7 \rightsquigarrow_\wedge 0.6) = (0, 0, 0.6) \\ s_3^- &= (0, a_3^- \rightsquigarrow_\wedge b, 0) = (0, 0, 0.8 \rightsquigarrow_\wedge 0.6) = (0, 0, 0.6) \end{aligned}$$

we have that

$$\begin{aligned} &\left(\bigcup_{j \in S^\oplus} [s_j^+, 1]\right) \cup \left(\bigcup_{j \in S^\ominus} [0, \neg s_j^-]\right) \\ &= ([(0,0.6,0),(1,1,1)] \cup [(0,0,0.6),(1,1,1)]) \cup [(0,0,0), \neg_S (0,0,0.6)] \\ &= ([(0,0.6,0),(1,1,1)] \cup [(0,0,0.6),(1,1,1)]) \cup [(0,0,0),(1,1,0.4)] \end{aligned}$$

Lastly, computing the intersection of the solution sets of Inequations (A1) and (A2), we obtain the solution set of Equation (A), that is:

$$[(0,0.6,0.4),(1,1,0.6)] \cup [(0,0,0.6),(1,1,0.6)] \cup [(0,0,0.4),(1,1,0.4)]$$

4 Conclusions and future work

We have studied the resolution of bipolar inequations in CDSRL with a join-irreducible element as independent term, providing the form of their complete solution set. This study has given rise to an alternative strategy in order to compute the solution set of bipolar equations in CDSRL.

As a future work, we are interested in addressing the solvability of bipolar equations in CDSRL whose independent term is a join-reducible element of the lattice, including the characterization of the solvability, the analytic form of the solution set and the algebraic structure of the solution set.

References

1. Birkhoff, G.: Lattice theory. American Mathematical Society, third edn. (1967).
2. Cignoli, R., Esteva, F.: Commutative integral bounded residuated lattices with an added involution. Annals of Pure and Applied Logic **161**(2), 150–160 (2009).
3. Cornejo, M.E., Lobo, D., Medina, J.: Bipolar fuzzy relation equations based on the product t-norm. In: 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). pp. 1–6 (2017).
4. Cornejo, M.E., Lobo, D., Medina, J.: Solving generalized equations with bounded variables and multiple residuated operators. Mathematics **8**(11), 1992 (2020).
5. Cornejo, M.E., Lobo, D., Medina, J.: Bipolar fuzzy relation equations systems based on the product t-norm. Mathematical Methods in the Applied Sciences **42**(17), 5779–5793 (2019).
6. Cornejo, M.E., Lobo, D., Medina, J.: On the solvability of bipolar max-product fuzzy relation equations with the product negation. Journal of Computational and Applied Mathematics **354**, 520–532 (2019).
7. Cornejo, M.E., Lobo, D., Medina, J.: On the solvability of bipolar max-product fuzzy relation equations with the standard negation. Fuzzy Sets and Systems **410**, 1–18 (2021).
8. Cornejo, M.E., Lobo, D., Medina, J., De Baets, B.: Bipolar equations on complete distributive symmetric residuated lattices: The case of a join-irreducible right-hand side. Fuzzy Sets and Systems **442**, 92–108 (2022).
9. Davey, B., Priestley, H.: Introduction to lattices and order. Cambridge University Press, second edn. (2002).
10. De Baets, B.: An order-theorethic approach to solving sup-T equations. In: Fuzzy Set Theory and Advanced Mathematical Applications, International Series in Intelligent Technologies, vol. 4, pp. 67–87. Springer US (1995).
11. De Baets, B.: Sup-T equations: state of the art. In: Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications, NATO ASI Series, vol. 162, pp. 80–93. Springer Berlin Heidelberg (1998).
12. Dilworth, R.P., Ward, M.: Residuated lattices. Transactions of the American Mathematical Society **45**, 335–354 (1939)
13. Drewniak, J.: Fuzzy relation equations and inequalities. Fuzzy Sets and Systems **14**(3), 237–247 (1984).
14. Freson, S., De Baets, B., De Meyer, H.: Linear optimization with bipolar max–min constraints. Information Sciences **234**, 3–15 (2013).
15. Grätzer, G.: General lattice theory. Birkhäuser Verlag (1998).

16. Hájek, P.: Metamathematics of fuzzy logic. Trends in Logic, Kluwer Academic Publishers (1998).
17. Li, P., Jin, Q.: On the resolution of bipolar max-min equations. Kybernetika **52**(4), 514–530 (2016).
18. Liu, C.C., Lur, Y.Y., Wu, Y.K.: Some properties of bipolar max-min fuzzy relational equations. In: Proceedings of the International MultiConference of Engineers and Computer Scientists 2015. Lecture Notes in Engineering and Computer Science, vol. 2222, pp. 955–969 (2015)
19. Liu, C.C., Lur, Y.Y., Wu, Y.K.: Linear optimization of bipolar fuzzy relational equations with max-łukasiewicz composition. Information Sciences **360**, 149–162 (2016).
20. Lobo, D., Cornejo, M.E., Medina, J.: Abductive reasoning in normal residuated logic programming via bipolar max-product fuzzy relation equations. In: 2019 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019). Atlantis Studies in Uncertainty Modelling, vol. 1, pp. 588–594. Atlantis Press (2019).
21. Sanchez, E.: Resolution of composite fuzzy relation equations. Information and Control **30**(1), 38–48 (1976).
22. Sanchez, E.: Inverses of fuzzy relations. application to possibility distributions and medical diagnosis. Fuzzy Sets and Systems **2**(1), 75–86 (1979).
23. Yang, X.P.: Resolution of bipolar fuzzy relation equations with max-łukasiewicz composition. Fuzzy Sets and Systems **397**, 41–60 (2020).
24. Zhou, J., Yu, Y., Liu, Y., Zhang, Y.: Solving nonlinear optimization problems with bipolar fuzzy relational equation constraints. Journal of Inequalities and Applications **2016**, 126 (2016).