Fuzziness and Lie Algebras

Giuseppe Filippone[1 [0000−0001−7315−1852]], Gianmarco La Rosa[1 [0000−0003−1047−5993]], Manuel Mancini[1 [0000−0003−2142−6193]], and Marco Elio Tabacchi[1,2 [0000−0001−7230−8192]]

[1] Dipartimento di Matematica e Informatica
Università degli Studi di Palermo, Italy
{giuseppe.filippone01,gianmarco.larosa,manuel.mancini,marcoelio.tabacchi}@unipa.it
[2] Istituto Nazionale di Ricerche Demopolis, Italy

Abstract. In this paper a very concise history of the relationship between fuzzy theory is given, from its origins to the fuzzification of algebraic structures, and Lie algebras, with special attention on exploring the use of fuzzy theory to generalise definitions and results of abstract algebra, and a focus on the fuzzification of Lie algebras. In order to reinforce the idea that strong links exists between the two disciplines, focus has been also put on the more recent and intriguing link between the two, namely the transfer principle, with suggestions on of this can be implemented in further research. The paper is meant as a starting point for renewing the discussion and fuelling further research on the topic.

Keywords: Fuzzy Logic · Lie Algebra · Fuzziness

1 Introduction

Since 1965, the work of L.A. Zadeh (1921-2017, see Figure 1 on the right) [27] has been a milestone in mathematical research, influencing various fields and contributing to the theoretical and applied development of all sciences (and not only). When he wrote his pioneering work, he was working at the University of Berkeley (he had been there since 1959). He became chair of the department in 1963, during which time he changed the department's name from the Department of Electrical Engineering (EE) to the Department of Electrical Engineering and Computer Science (EECS) [23]. To be fair, fuzzy theory has not been a success in the American scientific community. As an example, we quote the words of Professor William Kahan (University of Berkeley) in 1975:

> Fuzzy theory is wrong, wrong, and pernicious [...]. The danger of fuzzy theory is that it will encourage the sort of imprecise thinking that has brought us so much trouble. [19, p. 1]

However, as we will see in Section 2, fuzzy theory has influenced not only applied areas but also areas of abstract mathematics such as algebra. The contribution of Zadeh's work also lies in the strong inspiration it gave to mathematicians of the time and the developments that this inspiration led to. In this paper, we will therefore try to briefly summarise the role that fuzzy set theory has played in abstract algebra, starting from Rosenfeld's early results on groupoids and groups, and leading to its application in Lie theory (specifically in Lie algebras). Finally, we will discuss the *transfer principle*, a method that elegantly connects most classical results on crisp algebraic structures with their corresponding fuzzy counterparts. To achieve this, a small part of this paper will be devoted to the emergence of Lie theory, although we will refer the reader to much more detailed works on the history of mathematics.

In particular, in Section 2 we give a brief historical description of the fuzzification of algebraic concepts. In Section 3, we describe what a Lie algebra is and its fuzzy counterpart. In the next section, we describe the transfer principle — in the context of abstract algebra — that underlies the process of fuzzification of crisp concepts of abstract algebra. Finally, the concluding section will present a synthesis of the observations derived from the entire discussion.

2 Brief history of the fuzzification of basic algebra's concepts

The application of fuzzy set theory in abstract algebra can be certainly traced back to 1971 when Azriel Rosenfeld (1931–2004, see Figure 1 on the left) published a paper entitled "*Fuzzy Groups*" [21]. Although L.A. Zadeh is widely recognised as the father of fuzzy theory, "Rosenfeld is the father of fuzzy abstract algebra" [17, Preface, p. XII]. The article presents a fuzzy approach to generalise the definitions of algebraic structures of *groupoids* and groups, similar to Chin-Liang Chang's work in 1968 on topological spaces [4].

Fig. 1. Azriel Rosenfeld (1931–2004) and Lotfi Aliasker Zadeh (1921-2017)

In order to fully understand the context, it would be beneficial to quote the preface written by Rosenfeld in the book "*Fuzzy Commutative Algebra*" by John N. Mordeson and Davender S. Malik, published in 1998:

> The idea of trying to fuzzify algebra finally after several years later [from the first read of Zadeh's paper [27]], after Lotfi's student C.L. Chang had

> published his paper "Fuzzy topological spaces" (*J. Math. Anal. Appl.* 24 (1968) 182-190). After somewhat belatedly coming across that paper, I said to myself "If Chang can do it for topological spaces, I can do it for algebraic structures." [17, Foreword, p. X]

And he goes on to talk about the approach with which he wrote his article in 1971:

> I took a copy of my own algebra book with me on a train ride from Washington to New York, with the intention of formulating natural fuzzifications of the basic concepts of algebra. Needless to say, I found a way to do this; in fact, by the end of the train ride I had written an essentially complete draft of my embarrassingly well-cited paper "Fuzzy groups" (*J. Math. Anal. Appl.* 35 (1971) 512-517). [17, Foreword, p. X]

Now we will explore how to present the fundamental definitions of abstract algebra clearly and concisely, and what meaning we can derive from them.

A groupoid is an algebraic structure that requires very little: a closed binary operation on a set. We must define an operation that associates another element of the same set with a given pair of elements from the set. To simplify, we can refer to this operation as multiplication (and use multiplicative notation accordingly).

To be more precise, let G be a set and let $(\cdot)\colon G\times G \to G$ be the multiplication on G. The set G is a *groupoid* if $x\cdot y \in G$, for any $x, y \in G$. In essence, we require that the product of any two elements in the set must also be an element of the set. Before proceeding, it is necessary to note that the definition of a groupoid found in Rosenfeld's work is currently known as a *magma*. Today, the term groupoid refers to a different algebraic structure. Anyway, for historical reasons, we will continue to use the term groupoid in this article.

It is important to note the significance of this requirement, as it is easy to create binary operations that do not meet this criterion (as in the following example).

Example 1. Let $\mathbb{D}$ be the set of odd numbers and let $(+)$ be the usual addition. This operation is not closed in $\mathbb{D}$ since the sum of two odd numbers is even, hence it does not belong to $\mathbb{D}$.

The concept of belonging to a set is crucial, yet often overlooked. Zadeh's work raises questions about the degree of membership of a pair of elements that may not strictly belong to the set. Additionally, it prompts us to consider the implications for their product in such cases. As an example, we recall the definition of the fuzzy version of a *subgroupoid* as given by A. Rosenfeld in 1971. We do this for two reasons. The first lies in its historical relevance, as it is the first example in literature of the fuzzy version of a purely algebraic object. The second reason is theoretical. The essence of this definition lies in describing how a membership function behaves in relation to an operation introduced in a set. In other words, this definition describes the behaviour of the algebraic structure defined on a fuzzy (sub)set.

Definition 1 ([21]). *Let G be a groupoid. A fuzzy subset μ of G is called a* fuzzy subgroupoid *of G if, for all $x, y \in G$,*

$$\mu(xy) \geq \min(\mu(x), \mu(y)). \tag{1}$$

In other words, we require that the product of two elements belongs to the groupoid at least as much as the one that belongs to it less. Following Definition 1, we naturally encounter the classical definitions and results of abstract algebra, such as fuzzy subgroups and homomorphisms, which generalise their crisp versions. This method of generalising the closure of an operation can be applied to other algebraic structures that are much richer than groupoids. In abstract algebra, if we have an algebraic structure and want to define its natural substructures (such as group-subgroup, ring-subring, vector spaces-subspaces, etc.), the first requirement is that the operation inherited from the algebraic structure that contains it is closed. Hence, the condition expressed by Equation (1) will seem to define many of these algebraic structures and can be used to fuzzify any (sub-)algebraic structure.

3 Fuzzy Lie algebras

3.1 Origin of Lie algebras

Lie algebras can also be fuzzified using the same approach described in the previous section. They are algebraic structures with a geometric origin, which we will now summarise. To begin, we must answer a simple question: what is a geometric structure? An algebraic structure has a precise definition, it is a set endowed with an operation (unary, binary, and so on) that must satisfy some axioms. However, the same cannot be said for a geometric structure. Although we know that polygons, curves or surfaces are geometric objects, it is difficult (or impossible) to describe mathematically the subjective consideration linked to our experience or our sensations.

Felix C. Klein (1849-1925, see Figure 2 on the right) attempted to answer this question in its original paper [12] (see [11] for the corresponding translated paper). Specifically, a generalisation of geometry raises the following comprehensive problem:

> Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group. [11, p. 218]

Concurrently with Felix Klein's Erlanger Program, Marius Sophus Lie (1842-1899, see Figure 2 on the left) began studying smooth transformation groups, also known as continuous groups at the time. These groups were supported not only on a set but on a richer object that admitted a topology. This led to the definition of a *Lie group*, which is a smooth manifold equipped with a topology — that enables the application of methods from mathematical analysis — and endowed with a compatible group structure. In a Lie group, the group operation and the function associating each element with its inverse must be smooth functions. This means that they can be infinitely differentiated.

Fig. 2. Marius Sophus Lie (1842-1899) and Felix Christian Klein (1849-1925)

For a comprehensive historical account of Sophus Lie and Lie theory, we suggest consulting a more detailed and technical source (e.g., [8]).

Lie groups are not linear structures but rather curved manifolds. The power of Lie's theorem lies in simplifying the study of these objects by shifting their investigation to simpler ones. The basic idea, informally stated, is to associate a linear structure, primarily a vector space, with such complex objects as Lie groups, ensuring that this new structure inherits the group's properties. Furthermore, the results obtained on these linear structures should provide insight into the original Lie group. The solution to these questions can be found in *Lie algebras* (see Figure 3 for an example). They are vector spaces on which an operation, known as the *Lie bracket*, is defined. The Lie bracket must satisfy two properties. For clarity, we provide the definition here.

Definition 2. *Let* $\mathbb{F}$ *be a field. One has that* $\mathfrak{g}$ *is a* Lie algebra *over* $\mathbb{F}$ *if* $\mathfrak{g}$ *is an* $\mathbb{F}$*-vector space equipped with a bilinear form, known as* Lie bracket*, defined as follows:*

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

satisfying the following properties for all $x, y, z \in \mathfrak{g}$*:*

$$[x, x] = 0, \qquad \text{(alternating property)}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \qquad \text{(Jacobi identity)}$$

Lie's theorem simplifies many questions about Lie groups into questions about Lie algebras, that is, questions about smooth manifolds can be transformed into questions about linear algebra, resulting in a very wealthy theory.

It is important to note that Lie algebras, while having an abstract mathematical origin, are often studied purely as algebraic objects, disregarding their geometric roots. Despite their abstract construction, these structures have concrete applications in fields such as the theory of differential equations (e.g, [5, 18]), physics (e.g., [22, 25]), and robotics (e.g., [6]).

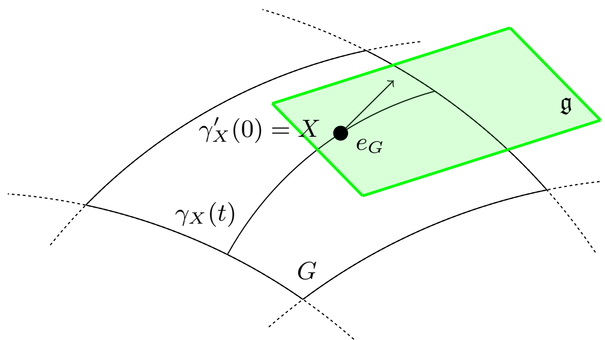

Fig. 3. The Lie algebra $\mathfrak{g}$ of the Lie group G.

3.2 How to fuzzify Lie algebras

A Lie algebra $\mathfrak{g}$, like any other algebraic structure, can be fuzzified in Rosenfeld style. A subalgebra of $\mathfrak{g}$, which is a subspace, remains closed with respect to the bracket operation. The definition of a fuzzy Lie subalgebra of a Lie algebra over a field was first introduced by Samy El-Badawy Yehia in [26]. Nevertheless, we use the notation in [2, Definitions 1.16 - 1.17] for our following definition.

Definition 3 (cf. [2]). *Let $\mathfrak{g}$ be a Lie algebra. A fuzzy set $\mu\colon \mathfrak{g} \to [0,1]$ is called a* fuzzy Lie subalgebra *of $\mathfrak{g}$ over a field $\mathbb{F}$ if*

1. $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
2. $\mu(\alpha x) \geq \mu(x)$,
3. $\mu([x,y]) \geq \min\{\mu(x), \mu(y)\}$,

for all $x, y \in \mathfrak{g}$ and $\alpha \in \mathbb{F}$.

In Lie algebras, as in other algebraic structures, it is possible to define richer subspaces that play a fundamental role in their study. These subspaces are known as *ideals*.

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. A subspace $\mathfrak{i} \subseteq \mathfrak{g}$ is called* ideal *of $\mathfrak{g}$ if $[x,y] \in \mathfrak{i}$, for any $x \in \mathfrak{i}$ and $y \in \mathfrak{g}$.*

Therefore, an ideal is a special subalgebra of a Lie algebra: not only is it closed under the Lie bracket operation, but if we compute this bracket between an element of the ideal and any element of the algebra, we still get an element of the ideal. Therefore, in this sense, we can think of ideals as Lie subalgebras that absorb (both from the left and from the right) the elements of the Lie algebra.

As we have seen for fuzzy subgroups, it is also possible to fuzzify the definition of *Lie ideals* (see, e.g., [2, Definition 1.18, p. 8]). Roughly speaking, for fuzzy Lie ideals one requires that the membership functions of the Lie bracket $[x, y]$, for any pairs $x, y \in \mathfrak{g}$, has to be at least equal to $\mu(x)$.

It is evident from this section that the class of Lie algebras is properly contained within the class of fuzzy Lie subalgebras. To obtain the original definition, one simply needs to consider the following membership function

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin \mathfrak{g} \\ 1 & \text{if } x \in \mathfrak{g} \end{cases}$$

This is not limited to Lie algebras but also applies to all fuzzified algebraic structures (groupoids, groups, vector spaces, and so on). Furthermore, as is customary when defining a generalised algebraic structure, a substantial portion of research is dedicated to finding analogous results for these new and more general structures. For instance, consider Levi's theorem for Lie algebras extended to Leibniz algebras (a generalisation of Lie algebras) [3]. From this perspective, it can be argued that many of the results that hold for crisp algebraic structures also hold for their fuzzy version, and vice versa. We will describe this principle in detail in the next section.

4 The Transfer Principle

In the field of model theory, the *transfer principle* posits that the validity of statements within a given language for one structure implies their validity for another structure within the same language.

Formally, the transfer principle allows one to transfer assertions from one algebraic system to another. The completeness of an elementary theory A, i.e., a collection of closed formulas of first-order predicate logic, implies a transfer principle for the models of A: every elementary sentence is true in all models of A if it is true in at least one model.

The forerunner of this principle was Gottfried W. Leibniz (1646-1716), who introduced an early version of it — although imprecise by present standards — referred to as "the Law of Continuity":

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the terminus may also be included. [10, pp. 902-903]

Leibniz's law was the precursor of the interpretation of the transfer principle that underlies modern infinitesimal calculus — although Leibniz presumed that infinitesimals possessed properties analogous to those of finite numbers.

The transfer principle can also be interpreted as a systematic formalisation of the *principle of permanence* (or the *law of the permanence of equivalent forms*), which asserts that algebraic operations such as addition and multiplication should demonstrate uniform behaviour across all number systems, particularly when expanding upon existing ones.

One of the applications of this method is in the field of unconventional analysis, which is known as *non-standard analysis.* In essence, the fundamental concept of this non-standard approach methodology in mathematical analysis is to construct a genuine arithmetic of infinitesimal numbers, thereby extending the real numbers $\mathbb{R}$ to the non-standard real numbers $^*\mathbb{R}$. In fact, this non-standard approach can be employed to extend any mathematical theory. In such a context, one of the main questions concerns the properties that are preserved (*transferred*) from $\mathbb{R}$ to $^*\mathbb{R}$, which ones are and which ones are not. It is unnecessary to provide a detailed account of the construction or its criticisms here. Readers are directed to more comprehensive and renowned readings for further information (see, e.g., [15] and [20]).

Michiro Kondo and Wieslaw Dudek in [13] applied the transfer principle in order to translate the results of crisp algebras to fuzzy algebras (for a schematic overview of the principle in question, please refer to Figure 4). Specifically, they divided the results of crisp algebraic structures into four types, and it was demonstrated that these types of propositions hold true for their fuzzy counterparts if and only if they hold true for their crisp versions. These types are the following:

type 0: A subset A has a property P;
type 1: If a subset A has a property P, then it has a property Q;
type 2: Let $f\colon X \to Y$ be a homomorphism. If a subset B of Y has a property Q, then a subset $f^{-1}(B)$ of X has a property P;
type 3: Let $f\colon X \to Y$ be a surjective homomorphism. If a subset A of X has a property P, then a subset $f(A)$ of Y has a property Q.

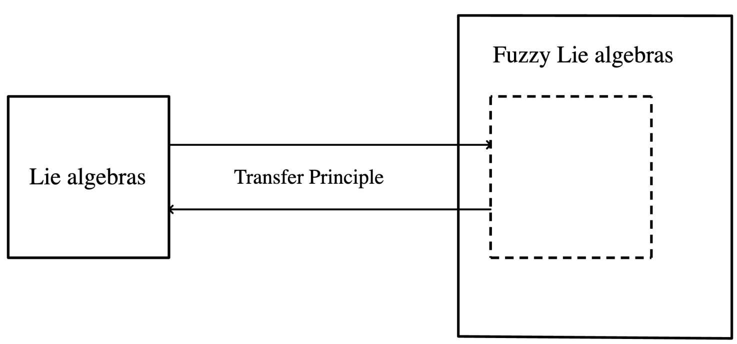

Fig. 4. Transfer principle for Lie algebras.

To be more precise, such application of the transfer principle began at least in 2003 in [9], a work by Young Bae Jun and Michiro Kondo. In this paper, the authors introduce this method and utilise it for fuzzy BCK/BCI-algebras. It is precisely in this context that they demonstrate how to extend certain concepts to their fuzzy version and immediately obtain numerous results in the fuzzy version, starting from crisp results. Hence, they showed that it is possible to match any crisp algebra substructure to its fuzzy counterpart, and vice versa. Their work led to some results in fuzzy Lie algebras such as those formalised in [2, 7]. Moreover, one can generalise for fuzzy Lie algebras the result in [2, Theorem 1.1, p. 8] by applying the transfer principle so far described. Here we show an example of this.

Theorem 1. *Every ideal of a Lie algebra is a Lie subalgebra.*

Theorem 2 (Proposition 1.4 [2]). *Every fuzzy Lie ideal is a fuzzy Lie subalgebra.*

In light of the findings presented in the work [13], it appears that the pursuit of analogous results to "crisp" algebraic structures for their fuzzified counterparts is an exercise in futility. However, this work also evidences that not all propositions and results can be classified into the four types showed above. In light of the aforementioned exposition and the accompanying historical observations, the next section will be devoted to the drawing of conclusions.

5 Conclusions

We have hitherto provided a concise overview of the history of the relationship between fuzzy theory, from its origins to the fuzzification of algebraic structures, and Lie algebras. Additionally, we described the more recent and intriguing element in this context, namely the transfer principle, in detail in Section 4. In particular, we provide some insight into the manner in which this principle has already been employed in diverse contexts, including model theory, non-standard analysis, and so forth. On the one hand, this principle offers a highly effective method for constructing a fuzzy version of every proposition known from the classical theory of Lie algebras, based on the crisp definition of a vector space, and vice versa. Conversely, since the definition of fuzzy Lie algebras extends that of the classic theory, some results can be appreciated in a natural way only in the fuzzy context, whereas they are irrelevant to the matter in the crisp Lie algebras (see, for example, [1, 14, 16, 24]).

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