Max Black and a Logic of Vagueness

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Abstract. A recent discovery by Rudolf Seising in the paper archives of Lotfi A. Zadeh highlights an epistolary correspondence between him and philosopher and fellow Bakuian Max Black on the theme of vagueness. In this paper, the article that Black suggests to Zadeh as a reading is discussed, along with its legacy in fuzziness.

Keywords: Fuzzy Logic · Vagueness · Loose Concepts.

1 Introduction

A discovery by Rudolf Seising in the paper archives of the father of fuzziness, i.e. Lotfi A. Zadeh, shed light on the fact that the then-still young researcher had sent his first articles on Fuzzy Sets to the already established philosopher, a dozen years older. Along with them, Zadeh acknowledges being indebted to the research carried out by Black on the search for a way to formally express the concept of vagueness, and probably above all for having inspired by his concept of Consistency Profile the methodology of Fuzzy Sets (see Figure 1). Black responds with the usual compliments reserved to a colleague and compatriot, but on this occasion points out to Zadeh some food for thought: the organisation of reasoning as the natural outcome of a Logic of Vagueness. This is, after all, one of the events that led to the birth of Fuzzy Logic, the term of which, it should be remembered, is not due to Zadeh himself. But an invention of George Lakoff [12, 14], derived from the work of Joseph A. Goguen, who had himself used "logic of inexact concepts" as a moniker for a logic of vagueness.

2 Max Black, a Logic of Vagueness

Black wrote the paper *Reasoning with Loose Concepts* [4] in 1962, and published it in 1963. It is a direct descendant of the renewed paper *Vagueness* [5], which focuses attention on how a well-known philosophical paradox attributed to Zeno is a tangible proof of how concepts are not sharp, and this should encourage to treat them as such.

Black deals in this paper with the problem of the sōrites paradox, as outlined in classical logic: how many elements must be removed from a group so that it

is no longer perceived as such. And, in a more abstract view, how strict the boundaries of a concept are, and how we deal with elements that are on the boundaries. A quotation from Cicero, which opens the article, outlines how the paradox itself is not at the heart of the problem, but instead, all attempts to affix quantifiers to predicates are. This is a two-thousand-year-old problem, to which non-classical logic will seek an answer in the twentieth century. In particular, Cicero notes how:

> [I]f we are asked by gradual stages, is such and such person a rich man or a poor man, famous or undistinguished, are yonder objects many or few, big or small, long or short, broad or narrow, we do not know at what point in the addition or subtraction to give a definite answer.

In this paragraph resides most of the sense of trying to convey the idea of quantity in fuzziness: any time a quantifier is used, we are introducing a form of gradation, and such gradation requires being able to deflect any yes or no answers to a more subtle, less defined framework. According to Black, the act of "drawing a line" between concepts is the one firing the sōrites, and the moment in which the categories of true and false are no longer applying [4, p. 2]:

> All these arguments depend upon gradual variation of some magnitude– the number of grains in a heap of corn, the number of hairs on a man's head, the length of a body, and so on–in a series of cases in which, as we commonly say, it seems hard to "draw a line." The fallacy dies hard. It is not uncommon to be faced with the peremptory demand, "Where do you draw a line?"; we are then confronted with the same sophism.

A plausible solution is to draw the line wherever one sees fit – especially if the one in question is in some sort of powerful position[3]. But such a line, whatever the partition it is implying and whoever is drawing it, will at some point feel arbitrary, and "unsatisfactory to anybody who has felt the force of the gradual progression" [4, p. 2].

In order to find an approach to "solve the sōrites", Black gives a more formal definition of the paradox, drawing from two of the most powerful classical logic tools: deduction and induction. The process of deconstructing the heap is now turned into a series of logical premises, of which the absurd outcome becomes the deductive conclusion. At the same time, the latent presence of numbers – which are not discrete by themselves, but whose arbitrary nature allows considering the small as it pleases ϵ as some sort of unitary increment – gives the deduction

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[3] The judge cited by Black in the following is reminiscent of justice Potter Stewart's: "I know it when I see it", and carries the same duck test bias that is so well described by William T. Goldberg in his essay on the different approaches to obscenity laws in British and US-bound culture [11]: "'I know it when I see it' can still be paraphrased and unpacked as 'I know it when I see it, and someone else will know it when they see it, but what they see and what they know may or may not be what I see and what I know, and that's okay'".

process an inductive feeling. Black himself defines as "*inductive*" the premise for which the addition of a small quantity to a known quantity does not modify its classification, and on this premise is focused on the possible explanation offered for the paradox.

When such premise is negated, i.e., when a line can be drawn between a concept and its negation due to the addition or subtraction of an arbitrarily small quantity to a known quantity, the relative concept is labelled as *sharply bounded*, while its opposite is *loosely bounded*. This nomenclature recalls Black's works of the thirties [5], and his lifelong interest in vagueness. But this time a sort of casual remark seems to shed light on how the term vagueness is not chosen for the boundary:

> A more common label, which I myself have used in the past, is "vague concept", but this may be misleading and has pejorative implications that I want to avoid. [4, p. 5]

There seems to be a hint of bitterness in this, as another instance of the well-known terminology vs. meaning debate in fuzziness. In light of this, it is not strange that Lakoff's choice and Zadeh's insistence on the term [14] is seen as an in-your-face choice that sharply contrasts with Black's apparent meekness.

Once the idea of a sharply and loosely based concept was defined, Black works back into incorporating it into the sōrites, and uses this to demonstrate the root of the paradox. In order to find a fault in the deduction, the only premise that can be attacked is the inductive one, as the other two are clear-cut and self-evident. But negating the inductive premise, and with that establishing the drawability of the cutting line discussed above, makes the concept dealt with sharply bounded. And this is exactly what the common knowledge denies about most of the empirical concepts. Black defines as empirical concepts the ones that possess at least one instantiation in the real world and goes as far as to say that "So far as I can see, all empirical concepts are 'loose'", expanding an idea already amply discussed twenty-five years earlier in [5], of which an already detailed description is given in [17]. The idea has even more remarkable precedence in Bertrand Russell [13] when he "propose[s] to prove that all language is vague". The distance with Russell's ideas, which was already wise in 1937 [17], has somehow widened in the meantime. Black redoubles on Russell's (and Gottlob Frege's) idea that vagueness can not be taken into consideration in pure logic, and connect the stance on every empirical concept being *vague* (here the word is used without shame, so no need to replace it with *loose*) to a sort of edict:

> The allegation that traditional logic is 'not applicable' to reasoning with loose concepts might mean that no question of logical validity can arise in connection with any such reasoning; or it might mean that reasoning with loose concepts, while sometimes correct, is not controlled by 'traditional' logical principles. [...] The resulting policy is tantamount to forbidding such reasoning. [...] Now if all empirical concepts are loose [...] the policy

becomes one of abstention from any reasoning from empirical premises. If this is a cure, it is one that kills the patient. A policy of prohibiting reasoning with loose concepts would destroy ordinary language – and, for that matter, any improvement upon ordinary language that we can imagine. [4, p. 6]

Furthermore, Black attacks the multivalued approach to the problem of vagueness. The addition of middle values, far from solving the conceptual problem, multiplies the line-drawing conundrum and the sharp boundaries between concepts.

How then to afford the sōrites in a way that takes into account the intrinsic vagueness[4] of concepts that overcome the Russel/Frege approach, multivalued logics and more in general overcomes the limits of "traditional" logic? The key to the attack lies in refusing the excluded middle principle. Frege connects the *tertium non datur* to the idea of a sharp concept and refuses to admit the existence of any loose one, as any that is not sharp is "wrongly termed a concept" [8][5].

The rejection of the excluded middle principle in the cases where concepts are not sharp brings to caution: "whenever we use logical principles in reasoning with loose concepts, we must be on the alert for gradual slides into indeterminacy [as] even the best-defined empirical concept eventually fails to discriminate – and there is nothing surprising about that" [4, p. 10]. It is the user of the concepts that should exert caution in using them, as there are cases in which even loose concepts can be used sharply. The presence of unsharp cases does not nullify the obviously sharp ones. The use of classical logic is fine when the demarcation between a concept and its contrary exists, but no rules exist for an automatic definition of this demarcation: a new logic is required for this.

The conclusion reached by Black in this paper is as follows: when concepts are loose – and all empirical concepts are loose to some degree – clear-cut cases or their use should be recognised as such and treated accordingly. Borderline cases should be dealt with by rejecting both the inductive premise and its contrary. This is not arbitrary, as the idea that one of the two options is unavoidable, according to the *tertium non datur* principle, is not written in the rules of our language:

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[4] "Our concept of being short is intended to be loose, is intended to have no sharp boundary. Anybody who thought it possible to find out the point of transition between short and not-short would be making a mistake about the present uses of the word 'short'. The question, 'How short is short" has no answer." [4, p. 5]

[5] More modern approaches to Frege's refusal to acknowledge the existence of vagueness in logic assign this stubbornness to the desire of understanding his "logicist project [...] as an attempt to establish the purely-logical status of ordinary arithmetical truths" and that "only under such an understanding can we take Frege at his word when he says that ordinary arithmetical sentences express truths and falsehoods." [6].

The man who presents [the sōrites] to us is trying to get us to use loose concepts as if they were strict; and our answer must be that we do not use our concepts in that way. [4, p. 11]

Loose concepts can be treated using logic, just not "*blindly*". In just a few years, a peek beyond the veil would be taken.

3   Later on, Vagueness in Fuzziness

While it is not explicitly known that Zadeh has followed Black's reading suggestions, it is often that he pays his respect to the original ideas in [5] (see, e.g., [2]). A hint of the inspiration that the 1963 paper has given to Zadeh is in the fact that it is cited in the bibliography of many articles, often without explicit it in the contents [20–23, 25, 26], or only as a reference in a footnote or appendix [18, 19, 24]. A probable sign that Zadeh considered this among the foundations that helped him formulate the idea of a logic for vague concepts.

In a recent volume on the history of fuzziness, Radim Bělohlávek et al. discuss the role of Black's formulation of the sōrites in Goguen's proposed fuzzy solution, attributing to [5] the antecedent in Goguen's research:

We now present the essence of a resolution of the sorites paradox using fuzzy logic which is due to Goguen (1968–69). One ingredient is the observation that the properties involved in sorites paradoxes, such as that of not forming a heap, are vague and are thus naturally modeled by fuzzy sets. This observation is essentially due to Black (1937). [3, p. 333]

In reality, it is the 1963 paper that constitutes the baseline for a fuzzy solution to the sōrites. Goguen [10] examines problems of *fuzziness* for "inexact concepts", where the word "concepts" is intended metaphorically. Specifically, in the first part, he formalized mathematically the Black's sōrites [4] concluding in accordance that:

It seems that we must abandon some crucial part of traditional logic, *modus ponens*, or else the Law of the Excluded Middle; or more precisely, that we are forced to abandon the idea that these parts of traditional logic apply to inexact concepts such as "short". [10, p. 330]

Goguen proposes a different representation for the word "short", through a fuzzy set (named J-set in the author's work). This representation led him to state:

[t]his model obviates the paradox. Moreover its properties correspond to intuitions about the original deductive process. In this sense, J-sets provide a good model for our use of "short". [10, p. 336]

Moreover, he describes how the three kinds of *inexactness*, i.e., *generality*, *ambiguity* and *vagueness* mentioned by Black [5] can be modelled as fuzzy sets. In particular, he states that

> generality occurs when the universe, or (depending on how you prefer to view things) the portion of the universe where degree of membership is large, is not just one point; ambiguity occurs when there is more than one local maximum of the membership function; and vagueness occurs when the function takes values other than just 0 and 1. [10, p. 344]

Renowned historian of fuzziness Rudolf Seising has included [4] in its omni-comprehensive volume on the history of fuzziness [16] and, in a *Festschrift* volume to Enric Trillas, both Seising and Senén Barro et al. have cited Black's paper [1, 15], but in both cases without any further information on how the contents where incorporated in Zadeh's future work. A critique to [4], due to James C. Bohan [7] and published in the same journal *Dialogue*, is based on the idea of concept relativity.

For a general reference to the history of fuzzy systems up to 1976, we address the paper of Brian R. Gaines [9], where he collected hundreds of bibliographic references about fuzzy system theory and its applications, many-valued logic, linguistic, and philosophy of vagueness.

Acknowledgements

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June 21, 1967

Professor L. A. Zadeh, Chairman
Department of Electrical Engineering
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Berkeley, California 94720

Dear Professor Zadeh,

 You were good enough to send me, some time ago, some of your
recent papers on topics connected with "Fuzzy Sets." If I have not
written before, the reason has not been lack of interest, but an
inescapable press of other duties.

 Now that I have had a chance, at last, to study your work, I
want to express my admiration and interest. I believe that your
ingenious construction promises to provide intellectual tools of
great value.

 In case you have not come across it, I might draw your attention
to an early article of mine, entitled "Vagueness" (Philosophy of
Science, Vol. 4, 427-455; reprinted in my book, Language and Philosophy,
Cornell University Press, 1949). A more recent article on similar
topics is "Reasoning with Loose Concepts" (Dialogue, Vol.2, June 1963,
1-12).

 I would be happy to see offprints of any of your further publica-
tions.

Yours sincerely,

Max Black

Max Black
Director

MB:sab
Enclosure

Fig. 1. A letter of acknowledgement, as well as encouragement, from Max Black to
Lotfi A. Zadeh; an answer to Zadeh's soliciting for an appraisal of his earlier work on
Fuzzy Sets (Archives of Rudolf Seising, published with kind permission).