Groupoids in categories of fuzzy topological spaces with continuous fuzzy relations

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Abstract. The notion of a continuous MV-valued fuzzy relation in Chang topological fuzzy spaces is defined, and the category **Top** of these spaces with continuous fuzzy relations as morphisms is presented. Two special subcategories of **Top** are presented, using the category of approximation spaces and the category of fuzzy partitions, both with fuzzy relations as morphisms. The concept of a fuzzy groupoid is defined for objects of these categories using the notion of fuzzy products in these subcategories.

Keywords: Chang L-fuzzy topological spaces; continuous fuzzy relation; fuzzy product in a category; fuzzy groupoid in a category

1 Introduction

Various extensions of classical algebraic structures to L-fuzzy structures are often used in the theory and applications of L-fuzzy sets. Typical examples are the extensions of monoids to L-fuzzy monoids [4, 5, 8], groups to L-fuzzy groups [3, 6] or rings to L-fuzzy rings [11, 15], and analogously for many other algebraic structures. Most of these L-fuzzy extensions use a standard procedure to define a L-fuzzy algebraic structure from a standard algebraic structure. If, for example, $S = (X, +)$ is an algebraic structure with one binary operation, then the L-fuzzy extension $\mathcal{S}$ of this structure can be defined as a L-fuzzy set $\mathcal{S} : X \to L$ such that for arbitrary elements $x, y \in X$, the inequality $\mathcal{S}(x + y) \geq \mathcal{S}(x) \wedge \mathcal{S}(y)$ is satisfied. You can proceed in an analogous way when expanding more complex algebraic structures. However, fuzzy extensions of algebraic structures defined in this way do not have the structure of classical algebraic structures on fuzzy objects, i.e. sets of fuzzy objects of a given type on which, for example, a binary operation is defined extending a classical binary operation in an underlying set. Fuzzy extensions of this second type can, for example, be obtained using various extension principles which extend binary operations to sets of fuzzy objects, such as the Zadeh extension principle or similar procedures. The disadvantage of this procedure is that the algebraic structures created in this way on fuzzy objects usually have very few properties similar to the properties of the original algebraic structures.

The problem of extending algebraic structures to fuzzy structures becomes even more complicated if the extended fuzzy structures are objects of the *category of fuzzy objects*, where the morphisms between objects are precisely defined. An example of such an extension can be the effort to extend the algebraic structure of a monoid to objects of different categories of fuzzy topological spaces, with differently defined morphisms between individual spaces. The complication of such a procedure then lies, among other things, in the fact that in order to extend, for example, a binary operation $+ : X \times X \to X$ to an object $\mathcal{X}$ of a category $\mathbf{K}$, we must ensure that in category $\mathbf{K}$ there is a certain type of product $\mathcal{X} \times \mathcal{X}$ of this object $\mathcal{X}$ and further that the extended new operation $\oplus : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a morphism in this category $\mathbf{K}$.

In category theory, there are tools that allow us to work with these extended algebraic structures. One such tool is the theory of monoids in monoidal categories (see [9]).This theory assumes that in a given category $\mathbf{K}$ there exists , among other things, a certain type of product $\boxplus : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$, which is a *functor* from the product of categories $\mathbf{K} \times \mathbf{K}$ to $\mathbf{K}$ and has some other relatively strict properties. A *monoid* in a monoidal category $\mathbf{K}$ is an object $\mathcal{X}$ of $\mathbf{K}$ with a morphism $\oplus : \mathcal{X} \boxplus \mathcal{X} \to \mathcal{X}$ that satisfies some axioms. However, these conditions and axioms can become difficult to fulfill, especially in a situation where $\mathbf{K}$ is a category of fuzzy objects, where the morphisms between objects are *relations* instead of classical mappings. This was mentioned, among other things, in [7], where a certain modification of these terms was also proposed.

Why can't these methods of category theory be directly applied to situations where morphisms are fuzzy relations? The problem lies in the definition of basic concepts in category theory, such as the product of objects. This concept requires, among other things, that for certain constructions there is exactly one morphism in $\mathbf{K}$ that satisfies the required properties, or that the so-called commutativity in the morphism diagram requires the exact equality of all paths in this diagram. Unfortunately, in the case of L-fuzzy relations as morphisms, these conditions are often impossible to satisfy.

In order to be able to use the methods of category theory for these categories as well, one possibility is to modify some categorical constructions and concepts so that they can also be used for these types of fuzzy set categories.

In this paper, we deal with two categories of L-fuzzy topological spaces, where the *morphisms are continuous L-fuzzy relations*. In these categories, which are based on the well-known category of L-fuzzy approximation spaces or the catagory of L-fuzzy partitions, we investigate conditions in which classical algebraic binary operations in sets can be extended to algebraic operations over objects of these categories in such a way that these extensions are again morphisms in these categories, i.e. continuous L-fuzzy relations. These conditions are presented as compatibility conditions between the original operation $+$ in the given set X and the structure of the object $\mathcal{X}$ of the category $\mathbf{K}$ extending object X.

2 Preliminaries

In the paper we deal with L-valued fuzzy sets, where L is a complete MV-algebra, i.e. algebraic system $(L, \vee, \wedge, \otimes, \oplus, \neg, 0_L, 1_L)$ such that $(L, \vee, \wedge, \otimes, 0_L, 1_L)$ is a complete residuated lattice and satisfies the following axioms:

$$\neg\neg x = x, x \oplus \neg 0_L = \neg 0_L, \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \neg(x \otimes y) = \neg x \oplus \neg y.$$

For more information on MV-algebras, see, e.g. [2]. By the L-fuzzy set in a set X we understand a mapping $X \to L$. One of the principal notions of the paper is the notion of the L-fuzzy relation. Unlike the standard notation of fuzzy relations from X to Y as fuzzy subsets in the Cartesian product $X \times Y$, we use a notation that comes from the theory of Kleisli categories of power set monads. For more information on these categories, see, e.g. [9, 16]. By a L-fuzzy relation from X to Y we understand a mapping $f : X \to L^Y$ that will be denoted by $f : X \rightsquigarrow Y$. If $g : Y \rightsquigarrow Z$ is L-fuzzy relation, its composition with f is indicated by $g\Diamond f : X \rightsquigarrow Z$ and is defined by

$$x \in X, z \in Z, \quad (g\Diamond f)(x)(z) = \bigvee_{y \in Y} f(x)(y) \otimes g(y)(z).$$

We can also use another definition of a composition $\Diamond$, denoted by $\Diamond^*$, defined by

$$x \in X, z \in Z, \quad (g\Diamond^* f)(x)(z) = \bigwedge_{y \in Y} f(x)(y) \oplus g(y)(z).$$

In [13], for arbitrary L-fuzzy relation $f : X \rightsquigarrow Y$ we introduced the following extending mappings $L^X \to L^Y$:

$$f^{\uparrow} := f\Diamond 1_{L^X}, \quad f^{\downarrow} := \neg f\Diamond^* 1_{L^X}$$

and "inverse" extending mapping $L^Y \to L^X$:

$$^{\Uparrow}f = f^{-1}\Diamond 1_{L^Y}, \quad ^{\Downarrow}f = \neg f\Diamond^* 1_{L^Y}.$$

These extended mappings have several important properties. Most of these properties were presented in [13]. We list some of these properties here for illustration.

Lemma 1. *Let $f : X \rightsquigarrow Y$ be a L-fuzzy relation and let $a \in L$, $s, p \in L^X, s_i \in L^X$, $r, t, t_i \in L^Y$, $i \in I$.*

1. $f^{\uparrow}(s) = \neg f^{\downarrow}(\neg s), \quad f^{\downarrow}(s) = \neg f^{\uparrow}(\neg s),$
2. $f^{\uparrow}(\bigcup_i s_i) = \bigcup_i f^{\uparrow}(s_i)$, $f^{\downarrow}(\bigcap_i s_i) = \bigcap_i f^{\downarrow}(s_i)$, $f^{\uparrow}(s \cap p) \le f^{\uparrow}(s) \cap f^{\uparrow}(p)$,
3. $^{\Downarrow}f(\bigcap_{i\in I} t_i) = \bigcap_{i\in I} {^{\Downarrow}f}(t_i)$, $^{\Uparrow}f(\bigcup_{i\in I} t_i) = \bigcup_{i\in I} {^{\Uparrow}f}(t_i)$,
4. $^{\Downarrow}f(a \oplus t) = a \oplus {^{\Downarrow}f}(t)$, $^{\Uparrow}f(a \otimes t) = a \otimes {^{\Uparrow}f}(t)$, $f^{\uparrow}(\alpha \otimes s) = \alpha \otimes f^{\uparrow}(s)$, $f^{\downarrow}(\alpha \oplus s) = \alpha \oplus f^{\downarrow}(s)$,
5. $^{\Uparrow}f(t) = \neg({^{\Downarrow}f}(\neg t)))$, $^{\Downarrow}f(t) = \neg({^{\Uparrow}f}(\neg t))$.
6. *If $f : X \rightsquigarrow X$ is reflexive, we have* $^{\Uparrow}f(t) \ge t \ge {^{\Downarrow}f}(t)$,
7. *Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be L-relations. We have*

$$^{\Uparrow}f.^{\Uparrow}g = {^{\Uparrow}(g\Diamond f)}, \quad ^{\Downarrow}f.^{\Uparrow}g = {^{\Downarrow}(g\Diamond f)}, \quad g^{\uparrow}.f^{\uparrow} = (g\Diamond f)^{\uparrow}, \quad g^{\downarrow}.f^{\downarrow} = (g\Diamond f)^{\downarrow}.$$

3 *L*-fuzzy topological spaces

In this section, we will define the basic properties of L-fuzzy topological spaces, and we present two examples of L-fuzzy topological spaces that we use in the rest of the paper. The basic definition of this type of space was presented by C.Chang [1], and a common feature of most publications on this space is that only continuous mappings are used for morphisms between two L-fuzzy topological spaces. In this section, we will therefore also focus on defining the concept of a continuous L-fuzzy relation.

We recall the definition of a strong Chang L-fuzzy topology.

Definition 1. *A strong Chang L-fuzzy topology in a set X is a subset $\mathcal{T} \subseteq L^X$ such that*

1. $\{s_i : i \in I\} \subseteq \mathcal{T} \Rightarrow \bigcup_{i\in I} s_i \in \mathcal{T}$,
2. $u, v \in \mathcal{T} \Rightarrow u \cap v \in \mathcal{T}$,
3. $\alpha \in L, s \in \mathcal{T} \Rightarrow \alpha_X \otimes s \in \mathcal{T}$
4. $\underline{0_L}, \underline{1_L} \in \mathcal{T}$.

Elements of $\mathcal{T}$ are open L-fuzzy sets and elements of $\mathcal{T}^c = \{\neg s : s \in \mathcal{T}\}$ are closed L-fuzzy sets. The pair $(X, \mathcal{T})$ is called a strong Chang L-fuzzy topological space, briefly L-fuzzy topological space only.

When defining the concept of a continuous L-fuzzy relation between two L-fuzzy topological spaces, we will proceed analogously as in the case of continuous mappings, with the difference that instead of a preimage of a mapping, we use a preimage of a L-fuzzy relation.

Definition 2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{F})$ be L-fuzzy topological spaces. A L-fuzzy relation $f : X \rightsquigarrow Y$ is called $\Uparrow$-continuous, if the implication holds:*

$$\forall t \in \mathcal{F} \Rightarrow {}^{\Uparrow}f(t) \in \mathcal{T}.$$

Now we can define the category of L-fuzzy tpologcal spaces with continuous L-fuzzy relations as morphisms.

Definition 3. *The category* **Top** *is defined by*

1. *Objects are L-fuzzy topological spaces $(X, \mathcal{T})$,*
2. *Morphisms from $(X, \mathcal{T})$ to $(Y, \mathcal{F})$ are $\Uparrow$-continuous L-fuzzy relations $f : X \rightsquigarrow Y$,*
3. *Composition of morphisms is defined by $\Diamond$,*
4. *For an object $(X, \mathcal{T})$ the unit morphism $1_{(X,\mathcal{T})}$ equals to $\eta_X : X \rightsquigarrow X$, defined by*

$$\eta_X(x)(x') = \begin{cases} 1_L, & x = x', \\ 0_L, & x \neq x' \end{cases}.$$

Examples of L-fuzzy topological spaces that we use in this paper are based on two types of L-fuzzy objects that are frequently used in fuzzy sets theory and applications. These objects are L-fuzzy approximation spaces and L-fuzzy partitions. For more information about these objects, see, e.g. [10, 12, 14, 17]. Unlike most publications dealing with these fuzzy objects, we will use L-fuzzy relations as morphisms in the categories of these objects.

Definition 4. *The category* $\mathbf{RSet}(L)$ *is defined by*

1. *Objects are L-fuzzy approximation spaces (X, δ), where $\delta : X \rightsquigarrow X$ is L-fuzzy similarity relation, that is, it satisfies the following axioms*
$$\delta \Diamond \delta \leq \delta, \quad \delta^{-1} = \delta, \quad \delta \geq \eta_X,$$
where $\eta_X : X \rightsquigarrow X$ is defined by $\eta_X(x)(x') = \begin{cases} 1_X, & x = x', \\ 0_L, & x \neq x' \end{cases}.$
2. *$f : (X, \delta) \to (Y, \gamma)$ is a morphism if $f : X \rightsquigarrow Y$ is L-fuzzy relation such that*
$$f \Diamond \delta \leq f, \quad \gamma \Diamond f \leq f.$$
3. *The composition of morphisms is defined by $\Diamond$.*
4. *The unit morphisms $1_{(X,\delta)}$ equal to δ.*

Analogically, we equip fuzzy partition objects with L-fuzzy relational morphisms.

Definition 5. *The category* $\mathbf{RSpace}(L)$ *of fuzzy partitions with L-fuzzy relational morphisms is defined by*

1. *Objects are pairs $[X, \mathcal{A}]$, where X is a set and $\mathcal{A} : |\mathcal{A}| \rightsquigarrow X$, where $|\mathcal{A}|$ is an index set and $\{core(\mathcal{A}(i)) : i \in |\mathcal{A}|\}$ is a partition of a set X, where $core(\mathcal{A}(i)) = \{x \in X : \mathcal{A}(i)(x) = 1_L\}$.*
2. *$(f, g) : [X, \mathcal{A}] \to [Y, \mathcal{B}]$ is a morphism if*
   (a) *$f : X \rightsquigarrow Y$ is a L-fuzzy relation,*
   (b) *$g : |\mathcal{A}| \rightsquigarrow |\mathcal{B}|$ is a L-fuzzy relation,*
   (c) *The following inequality holds:*
$$f \Diamond \mathcal{A} \leq \mathcal{B} \Diamond g.$$
3. *A composition of morphisms $(f, g) : [X, \mathcal{A}] \to [Y, \mathcal{B}]$ and $(f_1, g_1) : [Y, \mathcal{B}] \to [Z, \mathcal{C}]$ is a morphism $(f_1 \Diamond f, g_1 \Diamond g) : [X, \mathcal{A}] \to [Z, \mathcal{C}]$.*
4. *The unit morphism $1_{[X,\mathcal{A}]}$ equals $(\eta_X, \eta_{|\mathcal{A}|})$.*

By $\pi_{\mathcal{A}}$ we denote a mapping $X \to |\mathcal{A}|$ such that $x \in core(\mathcal{A}(\pi_{\mathcal{A}}(x))$. As follows from the following propositions, both these categories $\mathbf{RSpace}(L)$ and $\mathbf{RSet}(L)$ define subcategories in the category **Top**.

Proposition 1. *Let (X, δ) be an object of* $\mathbf{RSet}(L)$ *and let*
$$L^{(X,\delta)} = \{s \in L^X : \delta^{\uparrow}(s) \leq s\}.$$

1. $L^{(X,\delta)}$ *is the strong Chang L-fuzzy topology of* **open** *L-fuzzy sets in the set X. A L-fuzzy topological space* $(X, L^{(X,\delta)})$ *will be denoted by* $L(X,\delta)$,
2. $L_{(X,\delta)} = \{\neg s : s \in L^{(X,\delta)}\} = \{s \in L^X : \delta^{\downarrow}(s) \geq s\}$ *is the set of* **closed** *L-fuzzy sets in* $L(X,\delta)$.
3. *The full subcategory of* **Top** *with objects* $L(X,\delta)$ *will be denoted by* $\mathbf{Top}(L)$.

For objects of the category $\mathbf{RSpace}(L)$ we obtain another subcategory.

Proposition 2. *For an object* $[X,\mathcal{A}]$ *of the category* $\mathbf{RSpace}(L)$ *we set*

$$L^{[X,\mathcal{A}]} = \{s \in L^X : \forall i \in |\mathcal{A}|, \forall x \in core(\mathcal{A}(i)), \quad {}^{\Uparrow}\mathcal{A}(s)(i) \leq s(x)\}.$$

1. $L^{[X,\mathcal{A}]}$ *is a strong Chang L-fuzzy topology of* **open** *L-fuzzy sets in the set X. A L-fuzzy topological space* $(X, L^{[X,\mathcal{A}]})$ *will be denoted by* $L[X,\mathcal{A}]$.
2. $L_{[X,\mathcal{A}]} = \{s \in L^X : \forall i \in |\mathcal{A}|, \forall x \in core(\mathcal{A}(i)), \quad {}^{\Downarrow}\mathcal{A}(s)(i) \geq s(x)\}$ *is the set of* **closed** *L-fuzzy sets in* $L[X,\mathcal{A}]$.
3. *The full subcategory of* **Top** *with objects* $L[X,\mathcal{A}]$ *will be denoted by* $\mathbf{Top}(RSpace)$.

4 Topological groupoids in categories Top(L) and Top($RSpace$)

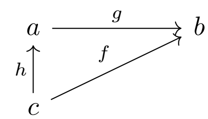
As we already mentioned in the Introduction, in order to be able to define the extension of the groupoid $(X,+)$ onto groupoids in these categories, we first of all need to define some type of products of objects in these categories and further compatible conditions allowing the binary operations + to be extended to morphisms in these categories.

Due to the properties of L-fuzzy relations, it is not possible to prove the existence of standardly defined products in categories (see, e.g., [9]) for any pair of objects in these categories. For this purpose, we will introduce the concept of L-fuzzy product in the category **Top**, which will allow us to introduce the concept of binary operation as a morphism in this category.

We show that the concept of a fuzzy product can be introduced in any category **K**, where the hom-sets $Hom_{\mathbf{K}}(a,b)$ of morphisms between two objects are endowed with order relations, preserving the composition of the morphisms. The fuzzy product represents a generalization of the product of objects in categories (see [9]), where the notion of commutativity of the diagram is replaced by the so-called fuzzy commutativity, and where instead of the existence of a *unique morphism* satisfying the given property, we demand the existence of the *largest morphism* with this property.

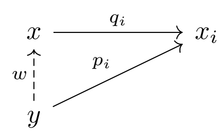
Definition 6. *Let* **K** *be an ordered category, that is, for arbitrary objects* x, y *of* **K**, *the hom-set* $Hom_{\mathbf{K}}(x,y)$ *of morphisms between* x, y *is ordered by a relation* $\leq$ *preserving the composition of the morphisms.*

1. *We say that the following diagram of* $\mathbf{K}$*-morphisms fuzzy commutes,*



if $g.h \leq f$ *holds.*

2. *Let* $\{x_i : i \in I\}$ *be a set of objects of* $\mathbf{K}$*. We say that* $x \in \mathbf{K}$ *with* $\mathbf{K}$*-morphisms* $q_i : x \to x_i$*,* $i \in I$*, is a fuzzy product of* $x_i, i \in I$*, if*
   (a) *For arbitrary object* $y \in \mathbf{K}$ *and* $\mathbf{K}$*-morphisms* $p_i : y \to x_i$*,* $i \in I$*, there exists a* $\mathbf{K}$*-morphism* $w : y \to x$ *such that the diagram fuzzy commutes,*



   that is, $q_i.w \leq p_i$*.*
   (b) w *is the largest* $\mathbf{K}$*-morphism* $y \to x$ *such the above diagram fuzzy commutes.*

The following theorem is an essential prerequisite for defining groupoids in categories of fuzzy objects.

Theorem 1. *In categories*

$$\mathbf{RSet}(L), \quad \mathbf{RSpace}(L), \quad \mathbf{Top}, \quad \mathbf{Top}(L), \quad \mathbf{Top}(RSpace)$$

there exist fuzzy products of arbitrary sets of objects.

Using the concept of a fuzzy product in an ordered category, we can now define the concept of a fuzzy groupoid in these categories. The advantage of the concept of fuzzy goupoid defined in this way is that its definition is formally analogous to the definition of a classical proupoid in sets.

Definition 7. *Let* $\mathbf{K}$ *be an ordered category, and let* x *be an object of* $\mathbf{K}$*. We say that* $G = (x, \oplus)$ *is a fuzzy groupoid in the category* $\mathbf{K}$*, if* $\oplus : x \times x \to x$ *is a morphism in* $\mathbf{K}$*, where* $x \times x$ *is a fuzzy product in* $\mathbf{K}$*.*

4.1 Fuzzy groupoids in category Top(*L*)

To extend a classical groupoid $(X, +)$ onto a fuzzy groupoid in a category $\mathbf{Top}(L)$ of L-fuzzy topological spaces $L(X, \delta)$ with continuous L-fuzzy relations, we first need to define the notion of compatibility between the groupoid $(X, +)$ and the L-fuzzy approximation space (X, δ). This compatibility is a generalization of a congruence relation.

Definition 8. *Let $S = (X, +)$ be a groupoid, and let (X, δ) be an object of* $\mathbf{RSet}(L)$*. We say that S is compatible with δ if for arbitrary $x_1, x_2, x'_1, x'_2, y \in X$ the following inequality holds:*

$$\delta(x_1 + x_2)(y) \otimes (\delta(x_1)(x'_1) \wedge \delta(x_2)(x'_2)) \leq \delta(x'_1 + x'_2)(y).$$

Using this compatibility condition we can extend a groupoid operation $+$ onto a morphism in the category $\mathbf{RSet}(L)$.

Proposition 3. *Let (X, δ) be an object of $\mathbf{RSet}(L)$ and let $S = (X, +)$ be a groupoid compatible with δ. Then a L-fuzzy relation $\oplus : X \times X \rightsquigarrow X$ defined by*

$$\mathbf{x} = (x_1, x_2) \in X \times X, y \in X, \quad \oplus(\mathbf{x})(y) = \delta(x_1 + x_2)(y)$$

is a morphism $\oplus : (X, \delta) \times (X, \delta) \to (X, \delta)$ in $\mathbf{RSet}(L)$ and $((X, \delta), \oplus)$ is a fuzzy groupoid in $\mathbf{RSet}(L)$.

Using the compatibility of $+$ and δ we can also obtain the following result.

Proposition 4. *Let $L(X, \delta)$ be an object of $\mathbf{Top}(L)$ and let $S = (X, +)$ be a groupoid compatible with δ. Then a L-fuzzy relation $\oplus : X \times X \rightsquigarrow X$ from Proposition 8 is a morphism $\oplus : L(X, \delta) \times L(X, \delta) \to L(X, \delta)$ in $\mathbf{Top}(L)$ and $(L(X, \delta), \oplus)$ is a fuzzy groupoid in $\mathbf{Top}(L)$.*

4.2 Fuzzy groupoids in category Top(*RSpace*)

To extend a classical grouopid $S = (X, +)$ onto a fuzzy groupoid in a category $\mathbf{Top}(Space)$ of L-fuzzy topological spaces $L[X, \mathcal{A}]$ with continuous L-fuzzy relations, analogously as in the previous subsection, we need first to define the notion of compatibility between a groupoid $(X, +)$ and L-fuzzy partition space $[X, \mathcal{A}]$.

We need the following lemma.

Lemma 2. *Let $[X, \mathcal{A}]$ be an object of category $\mathbf{RSpace}(L)$ and let $S = (S, +)$ be a groupoid in the category of sets with mappings. Let, for arbitrary $\mathbf{x}, \mathbf{x}' \in X \times X, y \in Y$, $+$ satisfy the following inequality,*

$$\rho_{\mathcal{A}}(x_1 + x_2)(y) \otimes (\rho_{\mathcal{A}}(x_1)(x'_1) \wedge \rho_{\mathcal{A}}(x_2)(x'_2)) \leq \rho_{\mathcal{A}}(x'_1 + sx'_2)(y),$$

where

$$\rho_{\mathcal{A}}(x, x') = \begin{cases} 1_L, & \exists i \in |\mathcal{A}|, x, x' \in core(\mathcal{A}(i)), \\ 0_L, & otherwise. \end{cases}$$

Then in the set $|\mathcal{A}|$ the binary operation $\boxplus$ can be defined by

$$i_1, i_2 \in |\mathcal{A}|, \quad i_1 \boxplus i_2 = \pi_{\mathcal{A}}(x_1 + x_2), \text{ where } \pi_{\mathcal{A}}(x_1) = i_1, \pi_{\mathcal{A}}(x_2) = i_1.$$

Using the operation $\boxplus$ and the L-fuzzy relation $\rho_{\mathcal{A}}$ we can define the notion of compatibility of an operation $+$ and L-fuzzy partition $\mathcal{A}$.

Definition 9. *Let $[X, \mathcal{A}]$ be an object of the category* $\mathbf{RSpace}(L)$ *and let $S = (S, +)$ be a groupoid in the category of sets. We say that S is compatible with a L-fuzzy partition $\mathcal{A}$ if for arbitrary $\mathbf{x}, \mathbf{x}', \mathbf{z} \in X \times X, y \in Y, \mathbf{i} \in |\mathcal{A}|^2$ the following inequalities hold:*

$$\rho_{\mathcal{A}}(x_1 + x_2)(y) \otimes (\rho_{\mathcal{A}}(x_1)(x_1') \wedge \rho_{\mathcal{A}}(x_2)(x_2')) \leq \rho_{\mathcal{A}}(x_1' + x_2')(y), \quad (1)$$
$$\mathcal{A}(i_1)(z_1) \wedge \mathcal{A}(i_2)(z_2) \leq \mathcal{A}(i_1 \boxplus i_2)(z_1 + z_2). \quad (2)$$

Using this compatibility, we extend the binary operation $+$ from a groupoid S to a morphism $(\oplus, \boxplus) : [X, \mathcal{A}] \times [X, \mathcal{A}] \to [X, \mathcal{A}]$ in category $\mathbf{RSpace}(L)$, where $\times$ is a fuzzy product in this category.

Proposition 5. *Let $[X, \mathcal{A}]$ be an object in* $\mathbf{RSpace}(L)$ *and let $S = (X, +)$ be a groupoid compatible with a fuzzy partition $\mathcal{A}$. Let $\oplus$ and $\boxplus$ be (crisp) L-fuzzy relations*

$$\oplus : X \times X \rightsquigarrow X, \quad \boxplus : |\mathcal{A}| \times |\mathcal{A}| \rightsquigarrow |\mathcal{A}|,$$

for $\mathbf{i} = (i_1, i_2) \in |\mathcal{A}| \times |\mathcal{A}|, j \in |\mathcal{A}|, \mathbf{x} \in X \times X$ defined by

$$\boxplus(\mathbf{i})(j) = \begin{cases} 1_L, & j = i_1 \boxplus i_2, \\ 0_L, & \textit{otherwise} \end{cases},$$
$$\oplus(\mathbf{x})(y) := \begin{cases} 1_L, & y = x_1 + x_2, \\ 0_L, & \textit{otherwise} \end{cases}.$$

Then $(\oplus, \boxplus) : [X, \mathcal{A}] \times [X, \mathcal{A}] \to [X, \mathcal{A}]$ is a morphism in category $\mathbf{RSpace}(L)$ *and $([X, \mathcal{A}], (\oplus, \boxplus))$ is a fuzzy groupoid in the category* $\mathbf{RSpace}(L)$.

The consequence of this proposition is the following proposition, showing that a standard groupoid $(X, +)$ compatible with a fuzzy partition $\mathcal{A}$ can be extended to a fuzzy groupoid in the category $\mathbf{Top}(RSpace)$.

Proposition 6. *Let $[X, \mathcal{A}]$ be an object in* $\mathbf{RSpace}(L)$ *and let $S = (X, +)$ be a groupoid compatible with a fuzzy partition $\mathcal{A}$. Then $\oplus : L[X, \mathcal{A}] \times L[X, \mathcal{A}] \to L[X, \mathcal{A}]$ is a morphism in* $\mathbf{Top}(RSpace)$ *and $(L[X, \mathcal{A}], \oplus)$ is a groupoid in category* $\mathbf{Top}(RSpace)$.

5 Conclusions

The purpose of the paper was to define the notion of a continuous L-fuzzy relation between arbitrary Chang L-fuzzy topological spaces, which can also be defined in categories other than the category **Set** of sets with mappings. As an example of such categories, in the paper we chose the category of fuzzy approximation spaces $\mathbf{RSet}(L)$ with relational L-fuzzy morphisms and the category of spaces with fuzzy partitions $\mathbf{RSpace}(L)$ with L-fuzzy relational morphisms. To illustrate the possibilities of Chang L-fuzzy topological spaces with continuous

L-fuzzy relations, we have shown how the classic groupoids in sets can be extended to groupoids in objects of these categories in such a way that groupoid operations are morphisms in these categories.

The paper represents the first introduction to the issue of continuous L-fuzzy relations defined in various categories with fuzzy objects of other categories. This concept allows, among other things, to define continuous algebraic operations as continuous L-fuzzy relations and to investigate the properties of these relational operations.

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