Visualization of Preference Matrices for Labeled Objects

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Abstract. Preference matrices are used to quantify the pairwise degrees of preference of each object over each other object. Visualization of preference matrices helps to understand the underlying preference structures, for example to identify the most and least preferred objects, groups of objects with similar preference patters, or inconsistencies in the preference structure. A recent method for visualizing preference matrices is PrefMap. Often objects are not only associated with pairwise preferences, but also with continuous or discrete labels such as price or size. In this paper we extend the PrefMap visualization method to data which possess such labels, which we call LPrefMap. LPrefMap allows to analyze how preference is affected by the corresponding object features. Experiments with two real world preference data sets indicate that LPrefMap is a very useful tool to visually gain valuable insights in the preference structure and to understand how the object features influence the preference.

Keywords: data visualization, preference data, object features

1 Introduction

Visualization helps users to understand data, using visual representations such as graphs, charts, or maps. This paper proposes a novel method for visual analytics of *pairwise preference* data, more specifically for the case that the objects have labels.

Pairwise preference data associate each pair of objects with a numerical value that quantifies the degree of preference of the first object over the second one. We distinguish *additive* (or *fuzzy*) preference [25, 6] where the values of each preference and its reverse preference sum up to one, and *multiplicative* preference [20] where the *products* of preferences and reverse preferences are equal to one. Multiplicative preferences p_m may be transformed to additive preferences p_a using the formula

$$p_a = \frac{1}{2}\left(1 + \frac{\log p_m}{\log \hat{p}_m}\right) \tag{1}$$

where $\hat{p}_m$ is the maximum multiplicative preference in this data set. An important property of preference relations is transitivity [7, 21, 22], and many types of transitivity have been defined for preference relations, for example generalized weak transitivity [14] or Einstein transitivity [19]. A transitive preference structure is also called *consistent*.

The number of pairwise preference values increases quadratically with the number of objects, which makes human pairwise preference ratings infeasible for large numbers of objects. Instead, humans are often asked to provide rank orders or utility quantifications of objects. Rank orders can then be mapped to *canonical* fuzzy preference relations [13], and utilities can be mapped to fuzzy preference relations [11], based on additive or multiplicative transitivity [12], or on Łukasiewicz Transitivity [5, 15]. Mapping preference relations to utilities is often not unique but may lead to upper and lower bounds which can be interpreted as interval type–2 membership values [18].

Recently, a visualization method called *PrefMap* [16] was proposed for the visualization of (additive) pairwise preference structures, similar to multidimensional scaling [4, 10, 23, 24]. In this paper we introduce a variant of PrefMap which can be used to visualize pairwise preference structures with labeled objects, and which we call *LPrefMap*. The paper is structured as follows: Section 2 quickly reviews the PrefMap method. Section 3 introduces the new LPrefMap method. Sections 4 and 5 present experiments with two real world data sets: the sushi and car preference data sets. Section 6 summarizes our conclusions.

2 PrefMap: Visualization of Preferences

In this section we briefly summarize the PrefMap algorithm as defined in [16]: The input to the algorithm is an $n \times n$ additive preference matrix P with elements $p_{ij} \in [0, 1]$ for all $i, j = 1, \ldots, n$ and $p_{ii} = 0.5$ on the main diagonal. First, we compute an $n \times n$ matrix X with elements

$$x_{ij} = p_{ij} - 0.5 \tag{2}$$

for all $i, j = 1, \ldots, n$. Then, we compute the eigenvectors $v_1, \ldots, v_n$ and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $X \cdot X^T$. We take the eigenvectors of the two largest eigenvalues, say v_1 and v_2, and compute the $n \times 2$ matrix Y containing the two–dimensional vectors $y_1, \ldots, y_n$, so that the first dimension of Y is $v_1 \cdot \sqrt{\lambda_1}$ and the second dimension of Y is $v_2 \cdot \sqrt{\lambda_2}$. Now we find the main direction of preference in Y as

$$r = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} (y_i - y_j) \tag{3}$$

and rotate Y as

$$y_k = y_k \cdot \begin{pmatrix} r_2 & r_1 \\ -r_1 & r_2 \end{pmatrix} \tag{4}$$

for all $k = 1, \ldots, n$. We find the main direction of preference in the rotated Y by computing r again (3), and if $r_1 < 0$, then we flip the horizontal components of Y

$$y_k = y_k \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

for all $k = 1, \ldots, n$. We create a scatter plot diagram of Y, which shows the nodes of the PrefMap. Finally, we connect all pairs of PrefMap nodes with two–colored lines, for example with black and grey, where the relative length of one color (for example grey) corresponds to the degree of preference for this pair of objects, and where the end in this color (grey) is connected to the node corresponding to the more preferred object. As a result, we obtain a diagram, where objects are arranged from top to bottom according to their preference ranks, and where collinearity of nodes corresponds to consistency of the preference structure. In the experimental sections we will show one data set (the sushi preference data set) for which we obtain a quite collinear PrefMap plot, which indicates a highly consistent preference structure, and another data set (the car preference data set) with a rather inconsistent preference structure.

3 LPrefMap: PrefMap for Labeled Objects

In this paper we introduce a variant of PrefMap for labeled objects, which we call *LPrefMap*. Each node in PrefMap corresponds to one object, and can be labeled by a unique identifier such as the object index (number) or the object name, such as type of sushi. PrefMap allows to analyze the preference structure and identify objects that are consistent or less consistent with the overall preference structure. In many applications, the objects not only have a unique identifier, but can also be associated with continuous or discrete labels, such as the price of a sushi (continuous) or the sushi type seafood, egg, meat, or vegetable (discrete). The idea of LPrefMap is to enhance the visualization of the nodes (objects) by the object features. For both continuous or discrete features this can be done by visualizing the nodes (objects) using symbols with different graphical features such as shapes, fill styles, colors, or sizes. In our experiments with the car data set we will present examples for this approach. If the preference structure is very consistent, then PrefMap will yield almost collinear nodes, so we may approximate the two–dimensional PrefMap by a one–dimensional PrefMap which only considers the vertical coordinate. This allows us to use the horizontal coordinate for the node (object) features. In our experiments with the sushi data set we will present examples for this approach. We will see that both approaches allow to visually interpret how preference is affected by the different (continuous or discrete) object features.

4 Experiments: Sushi Data Set

We illustrate the visualization of preference matrices with labeled objects using LPrefMap with two real world data sets: The sushi preference data set [8, 9] with continuous labels, and the car preference data set [1] with discrete labels.

The sushi data set [8, 9] contains data from questionnaire responses about different types of sushi. We use the data set 3a that considers 10 different types of sushi. 5000 persons were asked to sort these 10 different types of sushi according to their preference. We use this data set to construct a preference matrix P_{sushi}

in the following way: First, for each pair of objects (sushi types) we count how often the first one is preferred over the second one (higher rank given by the same person). Then for each pair of objects we divide the count of preference of the first one over the second one by the count of preference of the second one over the first one. We store these values in a matrix, set the main diagonal to one and obtain a multiplicative preference matrix. This multiplicative preference matrix is then converted to an additive preference matrix using (1), which yields

$$P_{\text{sushi}} = \begin{pmatrix} 0.5 & 0.511 & 0.494 & 0.562 & 0.576 & 0.519 & 0.637 & 0.256 & 0.54 & 0.791 \\ 0.489 & 0.5 & 0.48 & 0.545 & 0.562 & 0.509 & 0.627 & 0.24 & 0.527 & 0.761 \\ 0.506 & 0.52 & 0.5 & 0.565 & 0.578 & 0.524 & 0.642 & 0.257 & 0.547 & 0.791 \\ 0.438 & 0.455 & 0.435 & 0.5 & 0.516 & 0.46 & 0.571 & 0.197 & 0.481 & 0.727 \\ 0.424 & 0.438 & 0.422 & 0.484 & 0.5 & 0.448 & 0.549 & 0.178 & 0.463 & 0.689 \\ 0.481 & 0.491 & 0.476 & 0.54 & 0.552 & 0.5 & 0.613 & 0.232 & 0.52 & 0.765 \\ 0.363 & 0.373 & 0.358 & 0.429 & 0.451 & 0.387 & 0.5 & 0.132 & 0.408 & 0.652 \\ 0.744 & 0.76 & 0.743 & 0.803 & 0.822 & 0.768 & 0.868 & 0.5 & 0.788 & 1 \\ 0.46 & 0.473 & 0.453 & 0.519 & 0.537 & 0.48 & 0.592 & 0.212 & 0.5 & 0.74 \\ 0.209 & 0.239 & 0.209 & 0.273 & 0.311 & 0.235 & 0.348 & 0 & 0.26 & 0.5 \end{pmatrix} \quad (6)$$

Notice that due to the normalization in (1), each element of this preference matrix is in the unit interval $[0, 1]$, and more specifically at least one preference is 0 and at least one preference is 1. And as with every additive preference matrix, all elements of the main diagonal are equal to the neutral preference 0.5. In addition to the preference votes, the sushi data set contains the values of several features of each of the 10 types of sushi, from which we use the four continuous features: heaviness, eat frequency, price, and sales frequency. To make the features comparable, we normalize each feature to the unit interval.

Fig. 1 shows the PrefMap of the sushi preference data set. The ten objects (sushi) appear almost on a vertical line, which indicates a consistent (more specifically an additive transitive) preference structure. According to the preferences, the resulting rank order of the objects (from top to bottom) is "toro", "maguro", "ebi", "anago", "ikura", "tekka_maki", "ika", "uni", "tamago", "kappa_maki". The top object "toro" is quite distant from the second object "maguro", which indicates that "toro" is a very clear favorite. On the other end, the bottom object "kappa_maki" is also quite distant from the previous object "tamago", which indicates that "kappa_maki" is a very clear least favorite. Fig. 2 shows the LPrefMap of the sushi preference data set. Since the objects in the PrefMap are almost collinear, we only use the vertical coordinate and ignore the horizontal coordinate. This gives us the possibility to plot the continuous features across the horizontal coordinate. For the sushi data set, we have four continuous features: heaviness, eat frequency, price, and sales frequency. We normalize each of the features on the unit interval and display the four features as curves with different patterns (solid, dashed, dash–dotted, and dotted). The curve for feature "heaviness" (solid) is minimal for the most preferred object "toro" and maximal for the least preferred object "kappa_maki". The curves for the other three features "eat frequency" (dashed), "price" (dash–dotted), and "sales frequency" (dotted)

Fig. 1. PrefMap of the sushi preference data set. The objects are almost collinear, which indicates a consistent preference structure. The most preferred object is “toro” (top), and the least preferred object is “kappa_maki” (bottom).

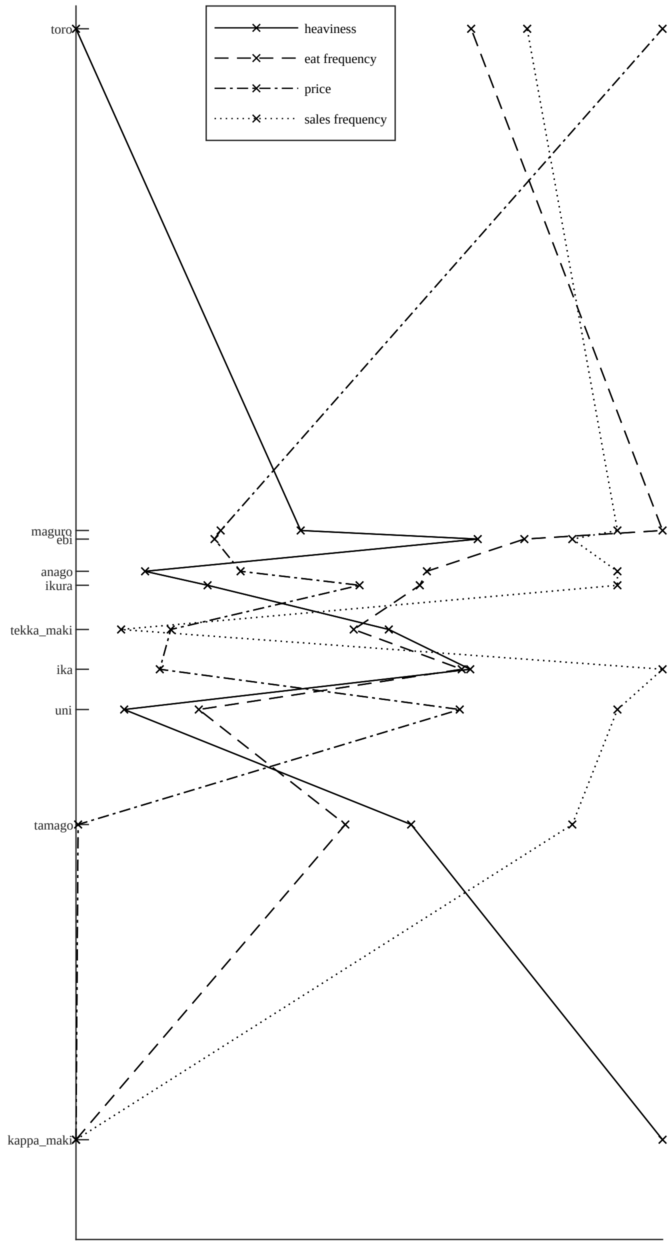

Fig. 2. LPrefMap of the sushi preference data set. The objects are almost collinear, so they can be visualized along the vertical axis. The four features are shown on the horizontal axis. The curve of the feature “heaviness” has a decreasing tendency, which indicates that preference is generally given for low heaviness. The curves of the other three features have increasing tendencies, which indicates that preference is generally given for high eat frequency, price, and sales frequency. The feature “heaviness” is partially compensated by the feature “price”: The objects “ebi” and “ika” have high heaviness, but are ranked relatively high because the price is low. Contrary, the objects “anago” and “uni” have low heaviness, but are ranked relatively low because the price is high.

show the opposite behavior: their feature values are high for "toro" and minimal for "kappa_maki". So in general preference seems to be given to sushi with low heaviness, high eat frequency, high price, and high sales frequency. However, for the eight objects with medium preference, none of the four curves shows a clear (increasing or decreasing) tendency but each curve oscillates between lower and higher values, so none of the four features alone is sufficient to predict the degree of preference. A closer examination reveals that the oscillations in the curves for heaviness and price are in opposite directions, so for the degree of preference, higher heaviness may be compensated by lower price, and higher price may be compensated by lower heaviness. This can be interpreted as follows: In general, preference is given to sushi with low heaviness, but sushi with higher heaviness may be given higher preference if the price is lower.

5 Experiments: Car Data Set

The car data set [1] contains data from user comparisons of pairs of cars. We use data from the first experiment that considers 10 different cars. Users were presented with a choice to prefer one of the 10 cars over another one of these based on their discrete attributes: body type (sedan or SUV), transmission (manual or automatic), engine capacity (2.5L, 3.5L, 4.5L, 5.5L, 6.2L), and fuel consumed (hybrid, non–hybrid). Random combinations of pairs of car features were presented to 60 users, and in each experiment the users decided for one of the two options. A total of 2973 experiments was performed. We use this data set to construct a preference matrix P_{car} in the same way as for the sushi data set: We count how often each object (car) is preferred over each other object (car), then divide the count of preference of each object over each other object by the reverse count, and set the main diagonal to one, which yields a multiplicative preference matrix. that we convert to an additive preference matrix using (1), which yields

$$P_{\mathrm{cars}} = \begin{pmatrix} 0.5 & 0.386 & 0.568 & 0.591 & 0.348 & 0.0638 & 0.127 & 0.0963 & 0.203 & 0.0638 \\ 0.614 & 0.5 & 0.788 & 0.865 & 0.779 & 0.465 & 0.685 & 0.455 & 0.676 & 0.477 \\ 0.432 & 0.212 & 0.5 & 0.488 & 0.386 & 0.104 & 0.135 & 0.164 & 0.239 & 0 \\ 0.409 & 0.135 & 0.512 & 0.5 & 0.315 & 0.185 & 0.185 & 0.299 & 0.256 & 0.157 \\ 0.652 & 0.221 & 0.614 & 0.685 & 0.5 & 0.21 & 0.265 & 0.185 & 0.396 & 0.193 \\ 0.936 & 0.535 & 0.896 & 0.815 & 0.79 & 0.5 & 0.642 & 0.807 & 0.71 & 0.633 \\ 0.873 & 0.315 & 0.865 & 0.815 & 0.735 & 0.358 & 0.5 & 0.455 & 0.735 & 0.488 \\ 0.904 & 0.545 & 0.836 & 0.701 & 0.815 & 0.193 & 0.545 & 0.5 & 0.676 & 0.558 \\ 0.797 & 0.324 & 0.761 & 0.744 & 0.604 & 0.29 & 0.265 & 0.324 & 0.5 & 0.351 \\ 0.936 & 0.523 & 1 & 0.843 & 0.807 & 0.367 & 0.512 & 0.442 & 0.649 & 0.5 \end{pmatrix} \quad (7)$$

Fig. 3 shows the PrefMap of the car preference data set. The most preferred object is number 6 (on top), followed by a group of four objects (10, 7, 8, 2), whose preference structure is quite inconsistent, leading to a horizontal spread among this group. Objects with medium preference are number 9 and number 5. The group of least preferred objects contains three objects (1, 3, 4, at the

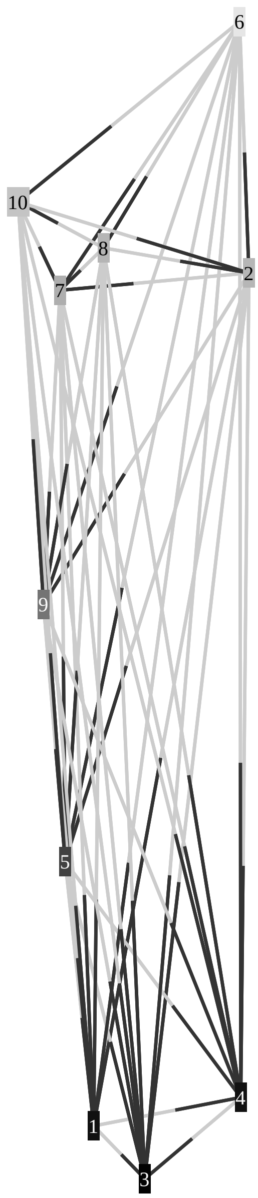

Fig. 3. PrefMap of the car preference data set. Object number 6 is most preferred, and objects number 1, 3, and 4 are least preferred. The preferences of objects number 10, 7, 8, and 2 show some inconsistency, as well as the preferences of objects number 1 and 4. The objects 10, 9, 5, 1 and the objects 6, 2, 4 are almost collinear, which indicates consistent preferences within these object groups.

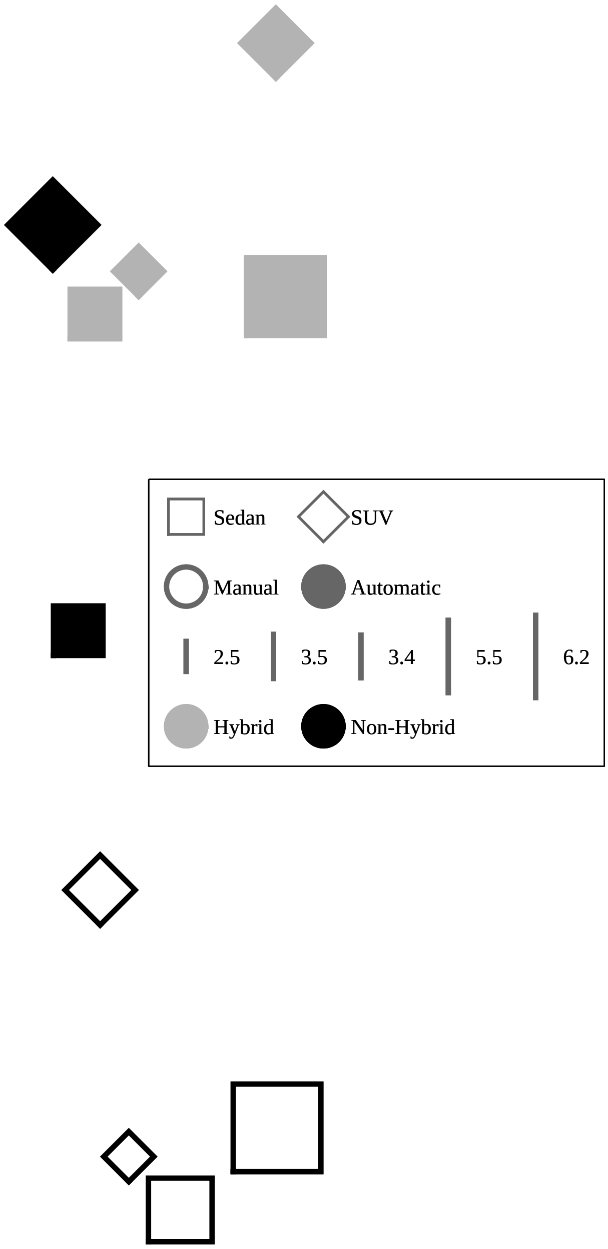

Fig. 4. LPrefMap of the car preference data set. The four discrete features are represented by different symbol shapes, fill styles, colors, and sizes. All 6 filled symbols are above the 4 unfilled symbols, which indicates a general preference of automatic over manual. All 4 grey symbols are above the 6 black symbols which indicates a preference of hybrid over non–hybrid, with one exception, a large non–hybrid automatic SUV at rank 2, but the preferences of this object are quite inconsistent with the preferences of the objects at ranks 3–5 (hybrid). Automatic cars are generally preferred over manual.

bottom), again with a quite inconsistent preference structure. The objects 10, 9, 5, 1 and the objects 6, 2, 4 are almost collinear, corresponding to a very consistent preference structure. Fig. 4 shows the LPrefMap of the car preference data set. The discrete attributes of the objects are visually represented by different shapes, fill styles, colors, and sizes, as shown in the figure legend: square for sedan and diamond for SUV, empty for manual and filled for automatic, sizes according to the engine capacity, grey for hybrid and black for non–hybrid. The LPrefMap reveals that hybrids (grey) are preferred over non–hybrids (black), except for a large non–hybrid automatic SUV (large black filled diamond) which is ranked number two. However, the preference structure among the objects of rank 2–4 are not very consistent, so there is disagreement for the high rank of the large non–hybrid automatic SUV. The LPrefMap also reveals that automatics (filled) are preferred over manuals (empty). The preference structures among the four smaller non–hybrids (black symbols on the left) and among the three larger engine capacities (larger symbols on the right) are very consistent.

6 Conclusions

Our experiments with the sushi and car preference data sets illustrate the capabilities of PrefMap and our new LPrefMap visualization method. PrefMap is a valuable tool for the identification of objects with high or low preference, and of inconsistencies in the preference structure. For the sushi data set, PrefMap (Fig. 1) indicates a very consistent preference structure and identifies the order of preference of the objects (sushi). For the car data set, PrefMap (Fig. 3) identifies objects (cars) with high and low preference and also detects inconsistencies in the preference structure. LPrefMap goes beyond this and allows to analyze how preference is affected by the different (continuous or discrete) object features. For the sushi data set, LPrefMap (Fig. 2) reveals that preference is generally given for objects (sushi) with high eat frequency, price, and sales frequency, and that the feature "heaviness" is partially compensated by the feature "price". For the car data set, LPrefMap (Fig. 4) clearly shows that automatic cars are preferred over manual, and hybrid cars are preferred over non–hybrid, with one exception, a large non–hybrid automatic SUV. These examples clearly indicate that LPrefMap is a very useful tool to visually gain valuable insights in the preference structure and to understand how the object features influence the preference.

Promising future research directions include the combination of LPrefMap with consensus visualization in group decision making [2, 3] or the extension to Pareto interval type–2 fuzzy decision making [17, 18].

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